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PLANE GEOMETRY

SLAUGHT AND LENNES

REVISED EDITION

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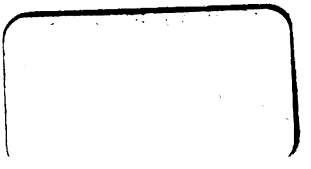

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Plato (429–348 B.C.) was an Athenian philosopher, who laid great stress on the remarkable relations between mathematics and philosophy. He made the study of geometry indispensable to that of philosophy (see page 2), and taught that the secret of the universe is to be found in *number* and *form*.

His greatest contribution to geometry was in emphasizing the need of formal definitions and axioms, and in developing the analytic method of discovering proofs. See page 107.

Above the entrance to his school of philosophy was the inscription:

"He who knows not geometry may not enter here."

PLANE GEOMETRY

WITH

PROBLEMS AND APPLICATIONS

REVISED EDITION

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PREFACE

THE movement for reform in the teaching of geometry now seems to have crystallized into certain rather definite conclusions. In rewriting this book the authors have sought to include the improvements resulting from this movement and at the same time to keep what was best in the older texts.

First among the improvements may be mentioned *practical applications* of geometry to everyday life. These are a feature of the new book. For example, see exercise 7 on page 161, a practical problem which should be brought to the notice of all interested in the game of football.

Among other improvements may be mentioned *informal proofs* of certain propositions which, though obvious even to the beginner, usage forbids stating as axioms; also, the relegating of difficult theoretical topics like *limits* to the end of the book, where the *incommensurable cases* are all treated together. In many respects the older treatment of limits is obscure and sometimes misleading. It is believed that the method here adopted is more defensible from every point of view.

The extreme *inductive method* of twenty years ago is introduced here only in a modified form. Theorems are proved completely, especially in the early part of the course. Propositions quoted in the proofs are at first given in full, later in part, and still later only by reference. After theorems have thus been quoted several times the pupil is left to his own devices in supplying the reasons.

Among the features of the older books which have been retained are the orderly and clear arrangement of the page and the simplicity of language in the definitions and proofs. The new book avoids technical and unusual language; the

term *equality*, for instance, is used consistently throughout the book instead of *congruence*. The word *congruent* has never made its way beyond the classroom; pupils and teachers alike speak of *equal* figures instead of *congruent* figures except during recitation hours. Moreover, much confusion has resulted from using the word "congruence" for triangles and polygons, and at the same time using "equality" for angles, arcs, and line-segments.

Perhaps the most important thing about the study of geometry for the beginner is the way in which he gets started. The authors, therefore, have given no small attention to the *Introduction*, — even extending it beyond the customary bounds. Particular attention is called to the illustrations of direct and indirect measurement (pages 24–28), which lead to the test of equality of two triangles by a course of reasoning, and which show the pupil both the need of theorems and demonstrations and also how to make a start in handling them.

An important feature of the new geometry is its *flexibility*. Many special topics may be omitted without disturbing the continuity of the course. Such topics are not involved in the exercises at the ends of the chapters. Of such nature is the work in Experimental Geometry, the numerous exercises in Sight Work, and the many Supplementary Topics. The main body of the book, when these various extras are excluded, consists of those theorems which have come to be regarded as constituting the minimum essentials of a standard course in geometry. The sight work and experimental geometry should provide welcome innovations to many progressive teachers. The supplementary topics and exercises will furnish plenty of work for the strongest secondary schools and will afford more than ample material for preparation for college examinations.

H. E. SLAUGHT
N. J. LENNES

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SUGGESTIONS TO STUDENTS

In entering upon the study of Geometry, the student may be assured that this subject has vital connection with many important phases of life in general, and that, in particular, it is capable of producing a profound impression upon his own mental development.

General Importance of Geometry. The material part of our civilization depends to a surprising extent upon that accurate information about space which is contained in geometry. Without geometry our machinery, our buildings, and our ships would be impossible. Our general mental outlook — our conception of the universe in which we live — depends upon geometry in no small degree. Without geometry we should know nothing of the magnitude of our earth as compared with the other planets, or of the distances separating the bodies of our solar system. See pages 26, 32, 175, 195, 204, 212, 214, etc.

Geometry in Engineering. The engineer uses geometric principles in making his structures secure as well as in making them pleasing to the eye. The principle of the rigidity of the triangle (page 42) is employed in securing the stability of bridges, buildings, and structures of all kinds. See any cyclopedia under "bridges." Look especially for "truss bridges" and "cantilever bridges." The computation made possible by formulas developed in geometry is of the utmost practical importance to the technical arts and industries. For the applications of some of these principles see pages 55, 113, 137, 217, 279, 281, 283, etc.

Geometry in Decorative Design. In designing a building an architect uses geometric figures to achieve beauty of form as well as to make the structure secure. The combinations of circles, and other geometric figures used in decorative designs are endless. In church windows, galleries, vaultings, mosaics,

tile patterns, they are found without number. For illustrations see pages 112, 116, 159, 162, 225, 278, etc.

Geometry and Mental Discipline. In the palmiest days of Greece, Plato inscribed above the entrance to his school of philosophy: "He who knows not geometry may not enter here." From that time to the present many of the great minds of the world have borne testimony to the value of the study of geometry as a mental discipline. It is said that Lincoln studied geometry at regular intervals, in order that his mind might become thoroughly accustomed to its rigorous logic. The student should be reminded that geometry is the only subject in the secondary schools in which he comes in contact with pure deductive logic. Some of the more difficult original problems contained in a text such as this one require a degree of ingenuity and perseverance that will form the student's best mental asset in after life. The worth-while problems of practical life are not such as may be solved at a glance or in a half hour. They require patience, originality, and clear-headedness. A thoroughgoing course in geometry should tend to develop these qualities better, perhaps, than any other course in the high school.

Geometry and Clearness of Expression. In the study of geometry the student learns to make clear and precise statements. In no other subject is he so certainly compelled to think correctly and to say precisely what he thinks. Clear thinking requires clear expression. Clearness of expression in turn reacts upon and clarifies the thinking. Clear thinking and clearness of expression may become fixed habits of the mind, and geometry is one of the most effective subjects for developing these mental habits.

The foregoing are some of the reasons why geometry should be studied by those who wish to obtain a generous culture, a broad outlook, and a mental development characterized by logical thinking and clear expression.

PLANE GEOMETRY

INTRODUCTION

Geometry is the science which deals with the properties of the space in which we live. That is, it treats of points, lines, surfaces, solids, and their position, form, and size.

The word "geometry," first used by the Greeks, is a combination of two words meaning "earth measurement" or "surveying." This name shows the early practical character of geometry.

1. **Geometry in Arithmetic.** In arithmetic many rules taken from geometry are used in the solution of problems. The areas of rectangles, triangles, and circles; the surfaces and volumes of rectangular solids, cylinders, pyramids, cones, and spheres, are all found by using *geometric rules or formulas.*

In arithmetic, however, the rules from geometry are only incidental, the main object of arithmetic being the study of *numbers* and *their application to the affairs of life.*

2. **Geometry in Algebra.** In algebra the main object is the study of the *equation* and *its uses in the solution of problems,* but the facts of geometry are sometimes used incidentally for illustration.

3. **The Main Object in Geometry.** In geometry the main object is the study of *geometric figures,* such as angles, triangles, parallelograms, circles, etc., and the development of a logical system of propositions about them. Definite proofs are made of the rules of geometry which have already been encountered in arithmetic and algebra; and many others are added which round out and complete the subject.

NOTE. — The student is advised to read now "Suggestions to Students" on the two preceding pages.

HISTORICAL NOTES

Geometry in Egypt and Babylonia. — The Egyptians appear to have been the first people to accumulate any considerable amount of information about geometry. A manuscript written by Ahmes, an Egyptian priest, sometime between the years 1700 and 1100 B.C., contains all the information about geometry which has come down to us directly from the Egyptians. This manuscript is believed to be an enlargement of an older treatise of about 3400 B.C.

The building of the great pyramids and of other marvelous structures (before 3000 B.C.) required considerable knowledge of geometry. The Egyptians also used geometry in surveying land. Thus, it is known that Rameses II (about 1400 B.C.) appointed surveyors to measure the amount of land washed away by the Nile, in order that the taxes might be adjusted.

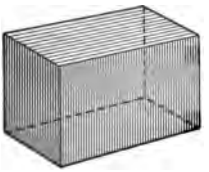
Inscriptions on tablets discovered in recent years have revealed some knowledge of practical geometry among the Babylonians.

Geometry in Greece. — While the Egyptians, and probably also the Babylonians, studied geometry for the purpose of using it in their practical work, the Greeks studied it for its own sake. The Egyptians knew only a few of the simplest facts of geometry, and there is no evidence that they ever made a definite proof of a single proposition. The Greeks, on the other hand, constructed a complete logical system, of which the facts derived from the Egyptians were the merest beginnings, and, moreover, they made complete proofs of all the propositions which they stated.

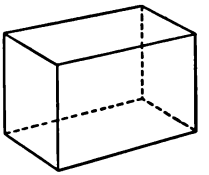
Thales. Pythagoras. Plato. — In about the year 600 B.C., Thales, a Greek from Asia Minor, learned geometry from the Egyptians, and began teaching it to the Greeks. Pythagoras (about 560 B.C.) discovered many new propositions and greatly increased the popularity of the study of geometry. From the time of Pythagoras to that of Plato, a succession of Greek mathematicians continued to make additions to geometry. Plato, himself, was one of the greatest of the ancient geometricians, and above the entrance to his school of philosophy was the inscription "*He who knows not geometry may not enter here.*"

Euclid's Elements. — About 300 B.C. Euclid, a Greek living in Alexandria, Egypt, wrote his "Elements of Geometry," in which he arranged in an orderly system all that was then known about geometry. This book contained the essential facts used in every text on elementary geometry which has been written since that time. It is safe to say that no book, save the Bible, has been printed so often, or studied so much, as Euclid's "Elements." For twenty centuries all civilized peoples have used it as a text in more or less modified form.

4. Elements of Space. The study of geometry begins with the consideration of certain elements of space, which are called *points, lines, planes, angles, triangles,* etc.



PHYSICAL SOLID



GEOMETRIC SOLID

5. Physical and Geometric Solids. Certain portions of space are occupied by objects which we call *physical solids,* as for instance a block of marble. The *space* occupied by a physical solid is called a *geometric solid.*

In geometry we are not concerned with the *matter* which occupies the space, but only with the *properties* of the space occupied, as for instance its shape and size.

Any limited portion of space is called a *geometric solid.* Such a solid is represented by the figure to the right above.

6. A Surface. That which separates a geometric solid from the surrounding space is called its *surface.*

The essential thing about a surface is that it has *no thickness,* though it has both length and width.

7. A Plane. If a surface is smooth and flat, we call it a *plane surface,* or simply a plane. A plane has *length* and *width,* but no *thickness.*

A sheet of paper is not a geometric plane, because it has some thickness. A gold leaf is more nearly a geometric plane because its thickness is much less than that of paper ($\frac{1}{250000}$ of an inch). If a gold leaf could be made thinner and thinner, and kept perfectly flat, it would approach more and more nearly to a geometric plane.

If a straight-edge is applied to a plane in any direction, every part of the edge will touch the plane. This is the general test as to whether or not a surface is a plane.

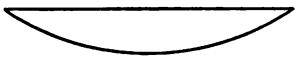
8. **Straight Line. Point.** Two adjoining surfaces of a solid like the one on page 3 meet in an edge which is a straight *line.* Two adjoining edges meet in a corner which is a *point.*

9. **Properties of a Straight Line.** A geometric line has *length,* but no *breadth* or *thickness.* If a line has the same *direction* throughout its entire extent it is a *straight* line.

The word *line* alone usually means straight line. A straight line is understood to be unlimited in length in both directions.

It is of course possible to think of a straight line without connecting it with the intersecting faces of a rectangular solid (§ 8). Thus, we speak of the *line of sight,* and we say that *light travels in a straight line.*

The term *straight* line is undoubtedly connected with the idea of a *stretched* string. Of all the lines which may be conceived as passing through two fixed points, that one is said to be straight which corresponds most nearly to a stretched string, such as a bowstring.



Likewise a plane may be thought of as straight or stretched in every direction, so that a straight line passing through any two of its points lies wholly in the plane.

That is, *if two points of a straight line lie in a plane, the whole line lies in the plane.*

It is assumed as evident that: *Through two points one and only one straight line can be drawn.*

This may also be stated: *Two points determine a straight line.*

SIGHT WORK

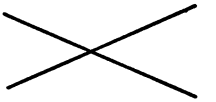
1. Use a straight-edged ruler to see whether the top of your desk is a plane surface.

2. In making a concrete sidewalk, how is the surface made smooth? Does this make it a plane?

3. How do the material lines made with crayon or pencil differ in magnitude from the ideal lines of geometry?

4. A machine has been made which rules 20,000 distinct lines side by side within the space of one inch. Do such lines have width? Are they geometrical lines?

10. Two Intersecting Straight Lines. If one of two intersecting straight lines turns about their common point as a pivot, the lines will continue to have only one point in common until, all at once, they will coincide throughout their whole length.



It is assumed as evident that: *Two straight lines cannot have more than one point in common.*

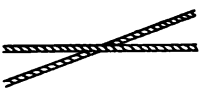
For if they have two points in common they coincide. See § 9.

This may also be stated: *If two straight lines have more than one point in common they coincide and are the same line.*

11. Properties of a Point. A geometric point has position but neither length, breadth, nor thickness. That is, a point has no extension and cannot be divided.

A small dot on the paper, used to represent a point, is not a geometric point since the dot has *length*, *breadth*, and some *thickness*. If the dot is made smaller and smaller, it corresponds more and more nearly to a geometric point. Such points are of course not material points but purely ideal, as are also geometric lines and planes.

SIGHT WORK



1. If a line had width as well as length, would the statements in italics in § 10 be true?

2. If two straight lines coincide in more than one point, what can you say of them throughout their whole length?

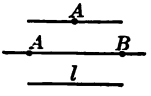
3. What is meant by saying: "Two straight lines determine a point"?

4. Why has a gun two sights on it?

5. A boy, in steering a boat across a lake, keeps the bow so directed that two objects on the other side of the lake appear to be in line with it. Is the boat going in a straight line? Explain.

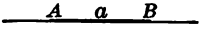
12. Notation for Points and Lines. A point is denoted by a capital letter.

A straight line is denoted by one small letter or by two capital letters marking two of its points.



13. A Line-Segment. That part of a line which lies between two of its points is called a *line-segment.*

A line-segment is designated by its end points or by a single letter, as the segment AB or the segment a.



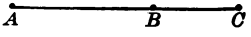
14. It is assumed as evident that:

(*a*) *A line-segment joining two points is the shortest distance between them.*

(*b*) *Two line-segments coincide and are equal if their end-points coincide.*

15. A Ray or Half Line. That part of an unlimited straight line which lies on one side of one of its points is called a *ray* or *half line,* as the ray AB.

16. Addition and Subtraction of Line-Segments. Two line-segments are said to be *added* if they are placed end to end so as to form a single segment.



Thus, segment AC = segment AB + segment BC, or $AC = AB + BC$.

If $AC = AB + BC$, then AC is greater than either AB or BC, and this is written $AC > AB$ and $AC > BC$. See § 72.

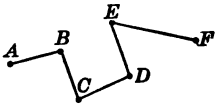
A segment may be *subtracted* from a greater segment or from an equal segment.

Thus, if $AC = AB + BC$, then $AB = AC - BC$ and $BC = AC - AB$.

17. Multiplication and Division of a Line-Segment. A segment is multiplied by an integer, n, by taking the sum of n such segments.

If AC is n times AB, then AB is an nth part of AC.

18. A Broken Line is composed of connected line-segments not all lying in the same straight line.



BROKEN LINE

CURVED LINE

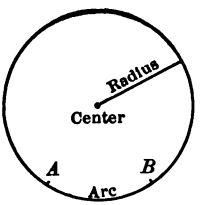
19. A Curved Line, or simply a **curve,** is a line no part of which is straight.

20. Geometric Figures. A point, line, surface, or solid, or any combination of these, forms a *geometric figure.* If all parts of a geometric figure lie in a plane it is a *plane geometric figure.*

21. Plane Geometry. Plane Geometry treats of plane geometric figures.

22. Closed Figures. A figure which entirely incloses a portion of the plane is called a *closed plane figure* or simply a *closed figure.*

23. Circle. Center. A circle is a closed plane curve all of whose points are equally distant from a fixed point in the plane called the center.



24. Radius. Any line-segment joining the center to a point on the circle is a *radius* of the circle.

25. Arc of a Circle. Any portion of a circle lying between two of its points is called an *arc,* as the arc *AB.*

26. It is assumed as evident that

(a) *A circle may be drawn with any center and any radius.*

(b) *All radii of the same circle or of equal circles are equal.*

Notice that in constructing a circle the distance between the legs of the compasses is not changed. Hence (b) is evident.

SIGHT WORK

1. Name some of the figures whose areas you found in arithmetic.

2. Name some of the solids whose volumes you found in arithmetic.

3. For what reason did the Egyptians study geometry? For what reason did the Greeks study it?

4. Name several surfaces which are plane or nearly so.

5. Name some smooth surfaces which are not plane.

6. If two points of a line lie in a plane, what can be said of the whole line?

7. Through two points how many straight lines can be drawn?

8. In how many points can two different straight lines meet?

9. What can be said of the magnitude of a point?

10. What can be said of the width and thickness of a straight line?

11. What can be said of the thickness of a plane?

12. How many end-points has an unlimited straight line? a line-segment? a ray? a circle?

13. Can you inclose a portion of a plane with two line-segments? With three? With four?

14. What can you say as to the equality of two radii of a circle?

15. Into how many parts will a point of a line divide it?

16. Into how many parts will a point of a circle divide it?

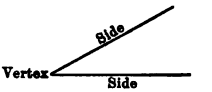
17. Into how many parts will two points of a circle divide it?

18. Into how many parts will two points of a line divide it?

19. Why is a taut string called straight? (§ 9.)

20. Why do people "cut across" a vacant lot?

27. An Angle. An angle is a figure formed by two rays, or segments, having a common end-point. The point is the *vertex* of the angle, and the rays, or segments, are its *sides*.

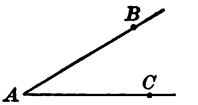


The *size* of an angle is determined by the *relative directions* of its sides, and not by their lengths.

An angle is sometimes defined as the *difference in direction* of two intersecting lines. The size of the angle is then thought of as the *amount of divergence* of the lines.

The hands of a clock form an angle which is constantly changing; the size of the angle increases as the difference in direction of the hands increases.

28. Notation for Angles. An angle is denoted by three letters, one at its vertex, and one marking a point on each of its sides.

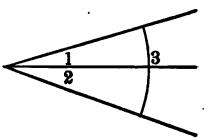


The letter at the vertex is read between the other two, as the angle *CAB*, or the angle *BAC*, not *ABC*. The letter at the vertex is also used alone to denote an angle in case no other angle has the same vertex, as, for instance, the angle *A*.

In written work the sign $\angle$ is used for the word angle.

Thus, we write $\angle A$ or $\angle BAC$, which is read *angle A* or *angle ABC*.

A small figure or letter placed within an angle is also a convenient notation for the angle. A small arc may be used in case of ambiguity.

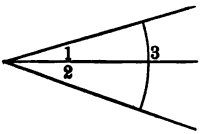


Thus, in the diagram $\angle 3$ includes $\angle 1$ and $\angle 2$, as shown by the arc near which the figure 3 is written.

29. Adjacent Angles. Two angles which have a common vertex and a common side between them are said to be *adjacent*.

Thus, $\angle 1$ and $\angle 2$ are adjacent angles. $\angle 1$ and $\angle 3$ have a common vertex and a common side, but they are not *adjacent* since the common side does not lie between them.

30. Adding and Subtracting Angles. The *sum of two angles* is the angle formed by their exterior sides when the two angles are placed adjacent.



Thus, $\angle 3 = \angle 1 + \angle 2$.

If $\angle 3 = \angle 1 + \angle 2$, then we say that $\angle 3$ is greater than either $\angle 1$ or $\angle 2$. This is written $\angle 3 > \angle 1$ and $\angle 3 > \angle 2$.

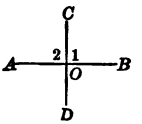
An angle may also be *subtracted* from a greater angle or from an equal angle.

Thus if $\angle 3 = \angle 1 + \angle 2$, then $\angle 3 - \angle 1 = \angle 2$ and $\angle 3 - \angle 2 = \angle 1$.

An angle may be *multiplied* or *divided* by a positive integer as in the case of a line-segment. See § 17.

However, an angle cannot always be divided into equal parts by *geometric construction*. See note, page 19.

31. Right Angle. Perpendicular. When one line meets another line so as to make two adjacent angles equal, each of these angles is a right angle and the lines are perpendicular to each other.

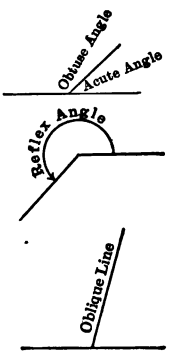


Thus, if $\angle 1 = \angle 2$, each angle is a right angle, and AB and CD are perpendicular to each other.

32. A Straight Angle. An angle is a *straight angle* if its sides lie in the same straight line and in *opposite* directions from the vertex.

Thus, in the figure of § 31, $\angle AOB$ is a straight angle.

33. Acute, Obtuse, and Reflex Angles. An *acute angle* is an angle less than a right angle.



An *obtuse angle* is an angle greater than a right angle and less than two right angles.

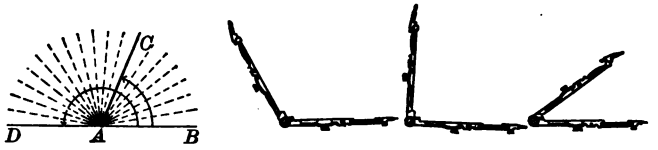
A *reflex angle* is an angle greater than two right angles, and less than four right angles.

Acute and obtuse angles are called *oblique* angles.

One line is *oblique to another* if the angles between them are oblique.

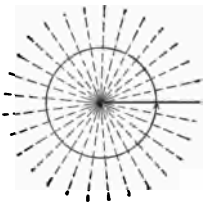
34. Generation of an Angle. A Perigon. An angle may be thought of as generated by a ray turning about its end-point as a pivot.

Thus $\angle BAC$ may be regarded as generated by a ray rotating from the position AB to the position AC.



If the ray continues to rotate until it lies in a direction exactly opposite to its original position, the angle generated is a *straight angle.*

If the ray rotates until it reaches its original position, the angle generated is called a *perigon.*



In elementary geometry angles greater than a straight angle are not considered.

35. Equality of Straight Angles and of Right Angles. It is assumed as evident that:

All straight angles are equal.

Since the straight angle AOB in the figure of § 31 is the sum of the two right angles, $\angle 1$ and $\angle 2$, it appears that *a straight angle equals two right angles, and that a right angle is half of a straight angle.*

It follows that:

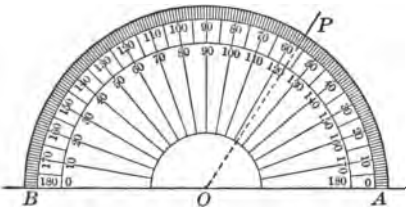
All right angles are equal.

36. Units of Measure for Angles. The units of measure for angles are *degrees, minutes,* and *seconds.* One three-hundred-sixtieth of a perigon is called a degree, one sixtieth of a degree is called a minute, and one sixtieth of a minute is called a second. These are denoted respectively by the symbols °, ′, ″.

Thus, we may speak of an angle of 20° 45′ 30″. A straight angle is an angle of 180°, and a perigon is an angle of 360°.

37. The Protractor. Angles may be measured by means of an instrument called a *protractor*, which consists of a semicircular scale with degrees from 0° to 180° marked upon it.

In the figure, the angle AOP is 57°.



An inexpensive protractor, made of cardboard or brass, may be had at any stationery store.

38. Vertical Angles. Two angles are called *vertical angles* if the sides of one are prolongations of the sides of the other.

Thus, in Fig. 1, $\angle 1$ and $\angle 3$ are vertical angles, as are also $\angle 2$ and $\angle 4$.

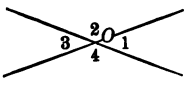


FIG. 1.

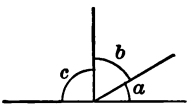


FIG. 2.

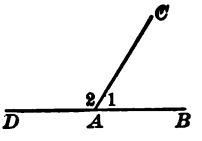


FIG. 3.

39. Complementary Angles. Two angles are said to be *complementary* if their sum is one right angle. Each is then called the *complement* of the other.

Thus, in Fig. 2, $\angle a$ and $\angle b$ are complementary angles.

40. Supplementary Angles. Two angles are said to be *supplementary* if their sum is two right angles. Each is then said to be the *supplement* of the other.

Thus, in Fig. 3, $\angle 1$ and $\angle 2$ are supplementary angles, since their sum is a straight angle and hence equal to two right angles.

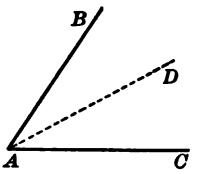
Two adjacent angles formed by one straight line meeting another are together equal to two right angles.

Such angles are called *supplementary adjacent angles.*

Thus, in Fig. 1, or Fig. 3, $\angle 1$ and $\angle 2$ are supplementary adjacent angles.

It is also evident that, *if two adjacent angles are supplementary their exterior sides lie in a straight line.*

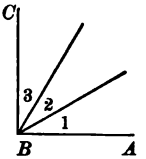
41. Bisector of an Angle. A ray which divides an angle into two equal angles is called its *bisector*.

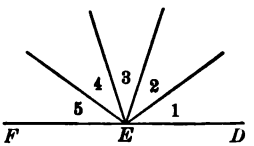


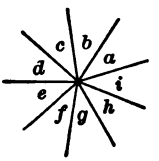
Thus, the ray AD bisects the angle BAC. In the figure of § 31, the perpendicular CD is the bisector of the straight angle AOB. See § 35.

SIGHT WORK

1. What part of a straight angle is a right angle?
2. How many degrees are there in a right angle?
3. If a right angle is bisected, how many degrees are there in each of the resulting angles?
4. Is an angle of 74° acute or obtuse?
5. Is an angle of 135° acute or obtuse?
6. Is an angle of 175° a reflex angle or an obtuse angle?
7. Is an angle of 192° an obtuse angle or a reflex angle?
8. Describe the angle made by the hands of a clock (*a*) when it is 2 o'clock; (*b*) 3 o'clock; (*c*) 4 o'clock.
9. Through what angle does the minute hand of a clock turn in 15 minutes? in 30 minutes? in 60 minutes?



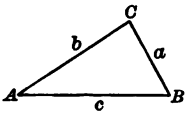




10. If $\angle ABC$ is a right angle and $\angle 1 = \angle 2 = \angle 3$, how many degrees are there in each of these angles?
11. If $\angle DEF$ is a straight angle and if $\angle 1 = \angle 2 = \angle 3 = \angle 4 = \angle 5$, how many degrees are there in each of these angles?
12. If, in the third figure, $\angle a = \angle b = \angle c = \angle d = \angle e = \angle f = \angle g = \angle h = \angle i$, how many degrees are there in each of these angles?
13. If a right angle is divided into six equal parts, how many degrees are there in each angle?

42. The Triangle and its Parts. A triangle is a plane figure bounded by three line-segments.

The segments are the *sides* of the triangle, and the points in which the sides meet are its *vertices*. The symbol $\triangle$ is used for the word "triangle." Thus we write $\triangle ABC$.



Usually small letters, as a, b, c, are used to represent sides, and the corresponding capital letters to represent the angles opposite these sides.

The three sides and three angles of a triangle are called the *parts of the triangle.*

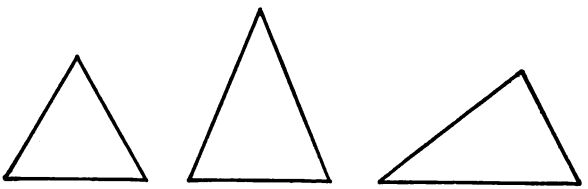
43. Opposite and Adjacent Parts of a Triangle. Each angle of a triangle has *one side opposite* and *two sides adjacent* to it. Similarly, each side of a triangle has *one angle opposite* and *two angles adjacent* to it.

Thus, in the above figure, the side a and the angle A are opposite parts, as are angle B and side b, and angle C and side c. The sides b and c are adjacent to $\angle A$, and the angles A and B are adjacent to the side c.

44. Included Parts of a Triangle. An angle of a triangle is said to be *included* between its two adjacent sides, and a side is said to be *included* between the two angles adjacent to it.

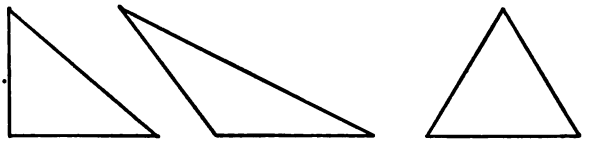
Thus, in the above figure, the side a is included between $\angle B$ and $\angle C$, and $\angle A$ is included between the sides b and c.

45. Triangles Classified by Sides. A triangle is called *equilateral* if the three sides are equal, *isosceles* if at least two sides are equal, and *scalene* if no two sides are equal.



EQUILATERAL ISOSCELES SCALENE

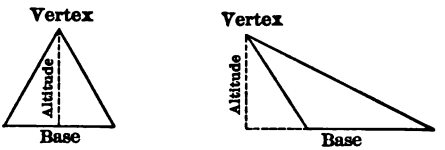
46. Triangles Classified by Angles. A triangle is called a *right* triangle if one angle is a right angle; *obtuse* if one angle is obtuse; *acute* if all angles are acute; and *equiangular* if all angles are equal.



RIGHT OBTUSE ACUTE

47. Hypotenuse and Legs of a Right Triangle. The side of a right triangle opposite the right angle is called the *hypotenuse* in distinction from the other two sides, which are sometimes called its *legs*.

48. Base and Vertex of a Triangle. The side of a triangle on which it is supposed to stand is called its *base*. The vertex of the angle opposite the base is called the *vertex* of the triangle.



49. Altitude of a Triangle. The *altitude* of a triangle is the perpendicular from the vertex to the base or to the base produced. Since any side of a triangle may be taken as its base, it follows that a triangle has three different altitudes.

SIGHT WORK

1. Is every equilateral triangle also isosceles? Is every isosceles triangle also equilateral?
2. Is a right triangle ever isosceles? Is an obtuse triangle ever isosceles? Draw figures to illustrate your answers.
3. Draw an acute triangle and its three altitudes.
4. Draw an obtuse triangle and its three altitudes.

50. Equal Figures. Direct Test of Equality. Two figures are said to be *equal* if, without changing the shape or size of either, they may be so placed as to coincide throughout. This is called *the direct test of equality*, or *the test by superposition.*

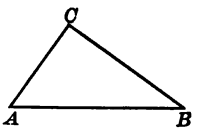
The equality of two figures is denoted by the symbol $=$, read *equals* or *is equal to.*

NOTE. — Equal figures are sometimes called *congruent*, that is, *filling the same space*, and the symbol $\cong$ is often used in place of $=$, but since congruence signifies *complete equality*, it seems best to say *equals* and to use the simpler symbol $=$.

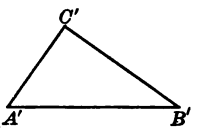
51. It is assumed that:

A figure may be moved about without changing its shape or size. This is called the axiom of motion. See Axiom 17, § 71.

52. Corresponding Parts of Equal Figures. If two triangles are equal then the equal sides are called *corresponding sides* and the equal angles are *corresponding angles.*



In the equal $\triangle ABC$ and $A'B'C'$, AB *corresponds* to $A'B'$, BC to $B'C'$, CA to $C'A'$, $\angle A$ to $\angle A'$, etc.



In general, corresponding parts of equal figures are parts which coincide when the figures are made to coincide.

Hence, we say that *corresponding parts of equal figures are equal.*

53. Indirect Test of Equality. The equality of the triangles ABC and $A'B'C'$, in § 52, may be tested by using tracing paper to draw a triangle $A''B''C''$ equal to $\triangle ABC$ and then applying it to $\triangle A'B'C'$.

Direct test by superposition is often inconvenient or even impossible. Thus, the equality of the triangles ABC and $A'B'C'$ could not be tested by *direct superposition* without cutting one out and applying it to the other.

Hence we say:

Two figures which are equal to the same figure are equal to each other.

EXERCISES

1. Find the complement of 34°; of 64°; of 29° 15′; of 39° 45′; of 78° 16′ 45″.

2. Find the supplement of 85°; of 64° 15′; of 108°; of 124° 15′ 15″; of 97° 20′ 40″.

3. The difference between two complementary angles is 21°. How many degrees are there in each angle?

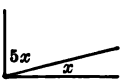
Suggestion: If x degrees is one angle, then $x + x + 21 = 90$.

4. The difference between two supplementary angles is 85°. How many degrees are there in each angle?

5. How many degrees are there in the smaller of the two angles formed by the hands of a clock at one o'clock? How many degrees are there in the complement of this angle? in the supplement?

6. How many degrees are there in the complement of an angle of 35°? in the supplement? What is the difference between the complement and the supplement of this angle? What is the difference between the complement and the supplement of any angle?

7. The complement of an angle equals five times the angle. How many degrees are there in the angle?



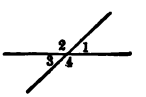
Suggestion: In the figure, $x + 5x = 90°$.

8. The complement of an angle is equal to $\frac{1}{8}$ of the angle. How many degrees are there in the angle?

9. The supplement of an angle is four times the angle. How many degrees are there in the angle?

10. The supplement of an angle is equal to $\frac{3}{4}$ of the angle. How many degrees are there in the angle?

11. If in the figure $\angle 1 = 45°$, find $\angle 2$. How many degrees are there in $\angle 2 + \angle 3$? Hence, how many degrees in $\angle 3$? How are $\angle 1$ and $\angle 3$ related?



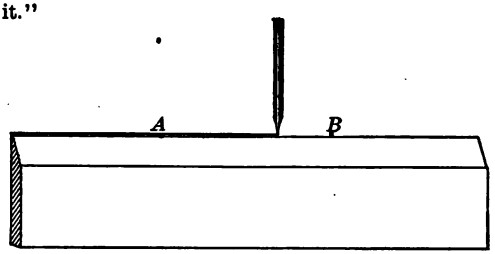
54. Instruments Used in Geometric Construction. In constructing geometric figures, the instruments used are the *straight-edge*, or ruler without division marks, and the *compasses*.

The custom of thus limiting the instruments used in *geometric construction* originated with the Greeks and has been universally followed. This limitation, however, is of theoretical rather than practical interest.

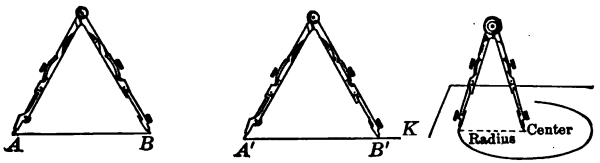
55. Use of the Ruler. By means of the ruler a straight line can be drawn through two given points.

To draw a line accurately through two points as A, B, the pencil should have a sharp chisel edge, and care should be taken to have the line pass exactly through the points.

The draftsman's rule is "Place the pencil on the point and move the ruler up to it."



56. Use of the Compasses. The *compasses*, or *dividers*, are used to construct a circle or an arc of a circle with a given radius and a given center, and also to lay off a segment on a given line equal to a given segment.



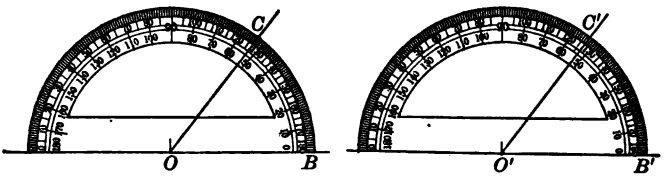
To lay off a segment equal to a given segment AB, place the points of the dividers on the end-points of the segment AB, and then transfer the dividers to the given line. Frequently one end-point, as A', of the required segment is given.

57. **Instruments used in Drawing Figures.** For the purpose of *drawing* figures, as distinguished from *constructing* them geometrically, other instruments besides the straight-edge and compasses are used.

The most important of these are: (*a*) the *ruler with division marks* for measuring straight lines, (*b*) the *protractor* for measuring angles, (*c*) the *triangle* for drawing right angles, and (*d*) the T-*square*. (See page 64.)

58. **Use of the Protractor.** The protractor may be used to draw an angle equal to a given angle.

To draw an angle equal to the angle BOC, we place the protractor on the angle BOC, as shown in the figure, and read the scale where OC crosses it.



Then draw another angle $B'O'C'$ using the center of the protractor as a vertex, drawing one side of the angle along the base of the protractor, and making the other side cross the scale at the same point as OC.

Later we shall show how to draw an angle equal to a given angle by using the ruler and compasses only.

The protractor may also be used to draw an angle of a given number of degrees. This is, in general, impossible by means of the ruler and compasses only.

EXERCISES

1. Using a protractor, decide whether or not, in the figure of § 52, $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

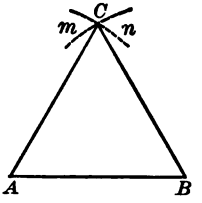
2. Draw any two intersecting lines and by use of the protractor find whether two pairs of equal angles are formed.

NOTE. — There are several classical problems in *geometric construction* which in recent times have been proved to be impossible. One of these is *the trisection of an angle*, which requires the division of any angle into three equal parts by means of the straight-edge and compasses only.

EXAMPLES INVOLVING THE USE OF INSTRUMENTS

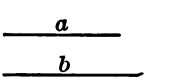
Example 1. *Construct an equilateral triangle.*

Solution: Draw a straight line and on it select two points A and B. With A as a center, and AB as a radius, strike an arc n. With B as a center, and AB as a radius, strike an arc m cutting arc n at the point C. Draw the segments AC and BC. Then $AC = BC = AB$ and $\triangle ABC$ is equilateral.

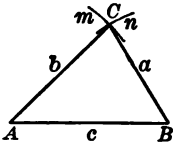


Example 2. *Construct a triangle with sides equal to three given segments.*

Solution: Let the given segments be a, b, c.



On a given line lay off segment AB equal to c. With A as a center and b as a radius, strike an arc m. With B as a center and a as a radius, strike an arc n, meeting arc m in C.

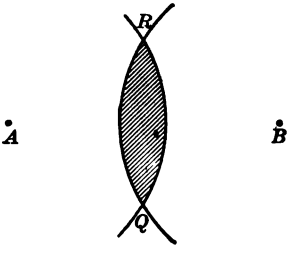


Draw segments AC and BC. Then $\triangle ABC$ is the triangle required.

In order that this construction may be possible, no one of the three given lines can be as long as the sum of the other two. See § 14 (*a*).

Example 3. Two horses are picketed 66 feet apart. Each horse has a picket rope which enables him to graze over a circle of radius 39 feet. Construct a figure showing the common area over which both can graze.

Solution: Let $\frac{1}{4}$ of an inch represent 6 feet. Then the 66 feet distance between the picket posts will be represented by $1\frac{3}{4}$ inches and the length of each rope by $\frac{1}{4}\frac{3}{8}$ of an inch. Let the picket posts be represented by A and B. With A as a center and a radius $\frac{1}{4}\frac{3}{8}$ of an inch strike an arc, and with B as a center and the same radius strike another arc. The area included between these arcs is the required area.

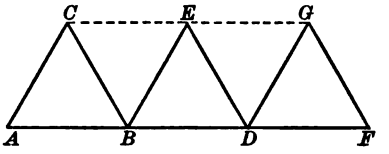


EXERCISES

1. Construct an equilateral triangle whose sides are each 2 inches.

2. Construct a triangle whose sides are 2 inches, $1\frac{1}{2}$ inches, and 1 inch.

3. Construct equilateral triangles as shown in the figure. Test with a ruler to see whether the points *C*, *E*, *G*, are in a straight line. Draw *CE* and *EG*, and measure to see whether triangles *CBE* and *EDG* are equilateral.



4. Two forts, 16 miles apart, are located on opposite sides of the entrance to a harbor. One fort has a range of 12 miles, and the other of 10 miles. Construct a figure to show the area which is within the range of both forts. Measure the figure to decide what is the shortest distance a hostile ship can sail under the fire of the forts. Let $\frac{1}{4}$ inch represent one mile.

5. A horse is picketed inside a square, fenced lot whose sides are each 60 feet. The rope is fastened successively in each of the four corners of the lot. Draw a figure representing the lot and the area over which the horse grazes if the rope permits the horse to graze over a circle of 30-foot radius.

Suggestion: Use the protractor to draw the angles of the square.

6. A horse is picketed outside the square, fenced lot of Example 5. Draw a figure representing the area over which he can graze if the rope is fastened successively at the corners of the lot.

7. In the preceding example suppose the rope to be fastened to the fence 20 feet from one corner. Draw a figure representing the area over which the horse will graze (*a*) if he is outside the fence, (*b*) if he is inside.

59. **Appearance of Figures Sometimes Misleading.** While many properties of geometric figures are evident on mere sight, we must not trust appearances too far.

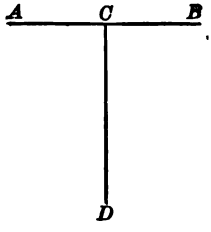


FIG. 1.

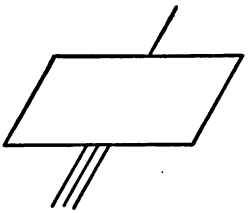


FIG. 2.

1. In Fig. 1, estimate which of the two segments AB and CD is the longer, and by how much. Measure to find how nearly right you are.

2. In Fig. 2, is the line above the figure a prolongation of one of the lines below it? If so, which one? How can you find out whether you are right?

3. Estimate which is longer, PQ or RS, and by how much. Measure to see how nearly right you are.

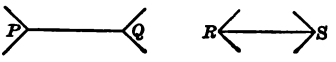


FIG. 3.

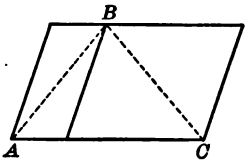


FIG. 4.

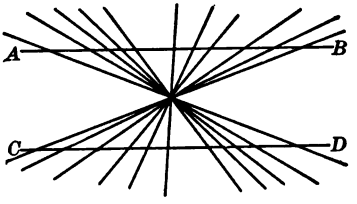


FIG. 5.

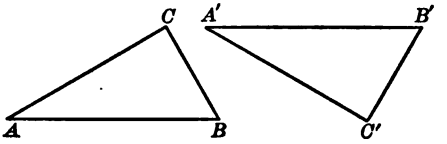
4. In Fig. 4, estimate which is longer, AB or BC, and by how much. Measure to see how nearly right you are.

5. In Fig. 5, are the lines AB and CD both straight lines? If not, which one is curved?

60. Direct Measurement not always Sufficient. To answer the questions on the opposite page, it was necessary to use one of the instruments described on pages 18, 19. Some of the questions below cannot be answered *even by use of these instruments.*

EXERCISES

1. In the triangles ABC and $A'B'C'$ are the following statements true:

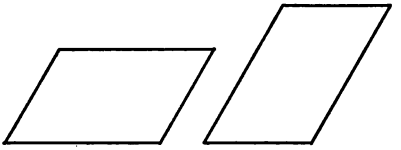


$AB = A'B'$,
$BC = B'C'$,
$CA = C'A'$,
$\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

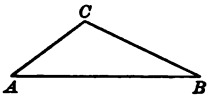
How can you answer each of these questions?

The question whether the triangles ABC and $A'B'C'$ are equal cannot be answered now by direct measurement alone. That is, the sides and the angles of the two triangles might conceivably be equal and the triangles still not be equal. It could be answered by using tracing paper as suggested in § 53.

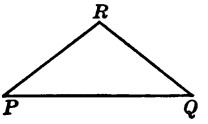
2. Using tracing paper decide whether the two figures in the margin are equal.



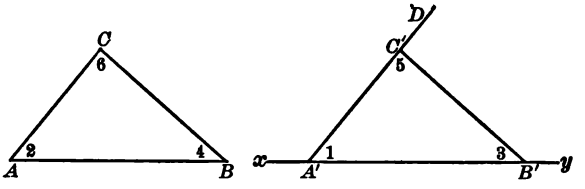
3. In the triangle ABC is $\angle A$ greater than $\angle B$? Can you decide this without the use of the protractor? If so, how? For another method see § 133.



4. In the triangle PQR are the angles P and Q equal? Can you decide this without using the protractor? If so, how? For another method see § 109.



61. Further Application of Direct Measurement. Given a triangle ABC. Draw a triangle $A'B'C'$ such that $A'B' = AB$, $A'C' = AC$, and $\angle A' = \angle A$, and compare the remaining sides and angles by direct measurement.



Solution. On a line xy, by use of the dividers, lay off $A'B' = AB$. By use of the protractor draw $A'D$ making $\angle 1 = \angle 2$.

On $A'D$ lay off $A'C' = AC$, and draw $C'B'$. Now, by measurement, find that $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, and $C'B' = CB$.

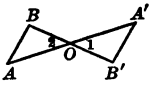
That is, we have found by measurement that if two triangles are drawn with two sides and the included angle of one equal, respectively, to two sides and the included angle of the other, the remaining pairs of corresponding parts are equal.

If we could prove that this statement is true in *every* case, we could then *measure three parts* in each triangle, namely, two sides and the included angle, and know that the remaining parts are equal *without further measurement.*

In the next article we shall find a method for making such a proof.

EXERCISES

1. Draw two intersecting lines as in the figure. Lay off $OA = OA'$, and $OB = OB'$. Find by measurement whether $AB = A'B'$, $\angle A = \angle A'$, $\angle B' = \angle B$, and $\angle 1 = \angle 2$.

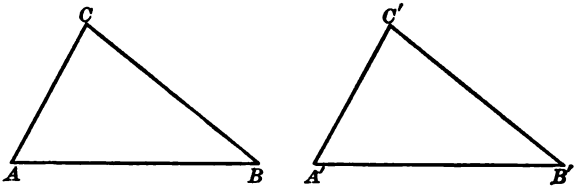


2. Draw two triangles ABC and $A'B'C'$ such that $AB = A'B' = 1$ inch, $AC = A'C' = 2$ inches, and $\angle A = \angle A' = 30°$. Then find by measurement whether $\angle B = \angle B'$, $\angle C = \angle C'$, and $BC = B'C'$.

62. Equality of Parts of Triangles shown by Argument. We now prove that in any two $\triangle ABC$ and $A'B'C'$,

$$\text{if } \begin{cases} AB = A'B', \\ AC = A'C', \\ \angle A = \angle A', \end{cases} \qquad \text{then } \begin{cases} BC = B'C', \\ \angle B = \angle B', \\ \angle C = \angle C'. \end{cases}$$

It will also be shown to follow that $\triangle ABC = \triangle A'B'C'$.



Argument:

Since $\angle A = \angle A'$, $\triangle ABC$ may be placed upon $\triangle A'B'C'$ so that $\angle A$ shall coincide with $\angle A'$, AB falling along $A'B'$, and AC falling along $A'C'$.

Since $AB = A'B'$, B will fall on B', and since $AC = A'C'$, C will fall on C'. Hence BC will coincide with $B'C'$ by § 14 (*b*).

Then $\angle B$ will coincide with $\angle B'$, and $\angle C$ with $\angle C'$.

We now know that $BC = B'C'$, $\angle B = \angle B'$, and $\angle C = \angle C'$ *without measuring these parts*, and, moreover, we have the further conclusion that $\triangle ABC = \triangle A'B'C'$, *since the argument shows that the triangles can be made to coincide throughout.*

We have, therefore, the following *general test* for the equality of triangles:

Two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

Notice that in the above argument *we do not actually move* $\triangle ABC$. It is sufficient to know that it *could be moved, and that the conclusion would follow absolutely if the triangle were moved.*

The argument used above may be called *argument by ideal superposition.*

63. Direct Measurement and Argument Contrasted. The use of argument in § 62 is superior to the direct measurement in § 61 in the following respects:

1. The *argument* not only showed that $\angle B = \angle B'$, $\angle C = \angle C'$, and $BC = B'C'$, which might be done by direct measurement, but also that $\triangle ABC = \triangle A'B'C'$, which it was impossible to show by *direct measurement.*

2. No measurement can be *absolutely exact;* hence any conclusion based on measurement is subject to possible error. But the test by argument is *absolutely exact;* that is, if it is granted that $\angle A = \angle A'$, $AB = A'B'$, and $AC = A'C'$ *exactly,* then $\angle B = \angle B'$, $\angle C = \angle C'$, and $BC = B'C'$ *exactly.*

3. The argument applies to *all* triangles in which the given conditions exist, while the test by measurement applies *only* to those triangles in which the unknown parts are actually measured.

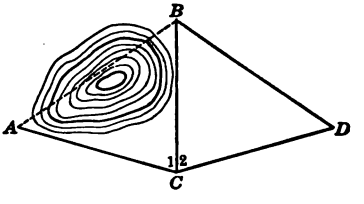
4. Direct measurement is often impossible.

64. Indirect Measurement. The combination of direct measurement with conclusions reached by argument or by general reasoning, which may be called *indirect measurement,* is of far-reaching importance. The following example affords a simple illustration.

Example. In surveying a farm it becomes necessary to find the straight line distance through a hill standing on a level plane. How can this be done?

Solution: Let A and B be two points on opposite sides of the hill, each being visible from a point C.

By means of an instrument for measuring angles, draw $\angle 2 = \angle 1$ and make $CD = AC$. Suppose BD drawn.



Then by § 62, $\triangle ABC = \triangle CBD$ and $BD = AB$.

Hence if we now measure BD, we shall find the length of AB.

Geometry the Basis of Indirect Measurement. In geometry an elaborate system of facts about the properties of space is built up, such, for instance, as the fact that two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other; and upon these facts are based nearly all indirect measurements.

Without geometry it would be practically impossible to dig a tunnel through a mountain by working from both sides, or to survey a mountainous landscape. We should be forever ignorant of the size of the earth, the distance to the moon or the sun, and many of our ideas of the universe would be as inaccurate and childish as were those of the ancients.

65. Reasoning in Geometry. The information gained in geometry about the properties of space, which could not be acquired by direct measurement, comes through *reasoning.*

The reasoning of geometry is much more thoroughly organized than any with which the student has thus far had experience, and we will now examine this kind of reasoning in some detail.

66. A Definition. A *definition* is a statement that a single word or phrase will be used in place of some more extended description of an object.

E.g. the word *circle* is used in place of the expression: "A closed plane curve all of whose points are equally distant from a fixed point in the plane called the center."

67. A geometric proposition is a problem to be solved or a statement affirming certain properties of a geometric figure.

Thus the statement: "If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, then the triangles are equal," is a geometric proposition; as is also the problem: "to construct a triangle equal to a given triangle."

68. A Proof. A proposition is *proved* or *demonstrated* when it is shown to follow from other known propositions.

69. A Theorem. A proposition to be proved is called a *theorem.* The argument used in establishing it is called a *proof.*

70. A Corollary. A proposition which follows easily from another proposition is called a *corollary.*

71. **Axioms.** Some propositions must be left unproved, since every proof depends upon other propositions. Propositions which for this reason are left unproved are called *axioms* or *assumptions.*

The axioms of geometry are of two kinds: (*a*) general axioms which apply as well to other branches of mathematics, particularly to arithmetic and algebra, and (*b*) axioms which apply to geometry only.

LIST OF AXIOMS

(A) **General Axioms which are applicable with certain restrictions to Geometry.**

1. *If equals are added to equals, the sums are equal.*

2. *If equals are subtracted from equals, the remainders are equal.*

3. *If equals are multiplied by equals, the products are equal.*

4. *If equals are divided by equals, the quotients are equal.*

5. *Quantities which are equal to the same quantity or to equal quantities are equal to each other.*

6. *The whole is equal to the sum of all its parts.*

7. *The whole is greater than any of its parts.*

8. *If a and b are quantities of the same kind, then $a = b$, $a < b$, or $a > b$.* See Abbreviations, § 72.

9. *If $a > b$ and $b \geqq c$, then $a > c$.*

10. *If $a \geqq b$ and $b > c$, then $a > c$.*

11. *If $a > b$ and $m = n$ (m and n being positive), then* $a + m > b + n$, $a - m > b - n$, $am > bn$, $\frac{a}{m} > \frac{b}{n}$.

12. *If $a > b$ and $c > d$, then $a + c > b + d$.*

13. *If $a = b$ and $c < d$, then $a - c > b - d$.*

14. *A quantity may be substituted for its equal in an equation or in an inequality.*

In plane geometry, the above axioms are applied to the lengths of line-segments and of curves, and to the magnitudes of angles and areas.

(B) **Axioms applicable to Geometry only.**

15. *Through two points one and only one straight line can be drawn.* See § 9.

16. *A line-segment joining two points is the shortest distance between them.* See § 14 (*a*).

17. *A figure may be moved about in space without changing its shape or size.* See § 51.

18. *A circle may be drawn with any point as center, and with any segment as radius.* See § 26 (*a*).

19. *All straight angles are equal.* See § 35.

20. *Two figures which are equal to the same figure are equal to each other.* See § 53.

It should be noted that axioms 5 and 20 are not equivalent. Axiom 5 refers to any quantities that have the *same magnitude* merely; while Axiom 20 refers to geometric figures that have both the *same magnitude and the same shape*. See § 50. Other axioms of geometry will be introduced as they are needed.

72. **Abbreviations.** The following symbols and abbreviations are used in geometry.

$\angle$, $\measuredangle$, angle, angles.
$\triangle$, $\blacktriangle$, triangle, triangles.
$\square$, $\boxed{7}$, parallelogram, parallelograms.
$\square$, $\boxed{8}$, rectangle, rectangles.
rt. $\angle$, rt. $\measuredangle$, right angle, right angles.
st. $\angle$, st. $\measuredangle$, straight angle, straight angles.
rt. $\triangle$, rt. $\blacktriangle$, right triangle, right triangles.
$\odot$, $\textcircled{9}$, circle, circles.
$\frown$, $\overset{\frown}{8}$, arc, arcs.
$=$, equals, *or* is equal to.
$\sim$, is similar to.
$>$, is greater than.
$<$, is less than.
$\leqq$, is less than or equal to.
$\geqq$, is greater than or equal to.
$\parallel$, parallel, or is parallel to.
$\perp$, perpendicular, or is perpendicular to.
$\parallel$s, $\perp$s, parallels, perpendiculars.
$\therefore$, therefore *or* hence.
Alt., alternate.
Ax., axiom.
Cor., corollary.
Def., definition.
Ext., exterior.
Hyp., by hypothesis.
Int., interior.
Th., theorem.

Q. E. D., *Quod erat demonstrandum*, which was to be demonstrated.
Q. E. F., *Quod erat faciendum*, which was to be done.

THEOREMS PROVED INFORMALLY

The following propositions which are assumed without formal proof are here collected for convenient reference.

73. *Two straight lines cannot have more than one point in common.* See § 10.

74. *All right angles are equal.* See § 35.

75. *Two line-segments coincide if their end-points coincide.* § 14 (*b*).

76. *Corresponding parts of equal figures are equal.* See § 52.

77. *Angles which are complements of the same angle or of equal angles are equal.*

For they are the remainders when the given equal angles are subtracted from equal right angles. See §§ 35, 39.

78. *Angles which are supplements of the same angle or of equal angles are equal.*

For they are the remainders when given equal angles are subtracted from equal straight angles. See §§ 35, 40.

79. *Vertical angles are equal.*

They are supplements of the same angle. See § 38.

80. *If one of two supplementary angles is less than a right angle, then the other is greater than a right angle.*

For if $\angle A + \angle B = 2$ rt. $\measuredangle$,
and $\angle B < 1$ rt. $\angle$,
then $\angle A > 1$ rt. $\angle$, by Axiom 13, § 71.

81. *Two adjacent angles formed by one straight line meeting another are together equal to two right angles.* See § 40.

82. *If two adjacent angles are supplementary their exterior sides lie in a straight line.* See § 40.

83. *The sum of all consecutive angles about a point in a straight line and on one side of it is two right angles.* See Axiom 6, § 71.

84. *The sum of all consecutive angles about a point in a plane is four right angles.* See Axiom 6, § 71.

85. *All radii of the same circle or of equal circles are equal.* See § 26 (*b*).

DEFINITIONS

The following definitions are collected here for convenient reference:

86. *A line-segment is that part of a line which lies between two of its points.* § 13.

87. *A ray is that part of an unlimited straight line which lies on one side of one of the points of the line.* § 15.

88. *A point, line, or surface, or any combination of these, all lying in one plane, forms a plane geometric figure.* § 20.

89. *A circle is a closed plane curve all of whose points are equally distant from a fixed point in the plane called the center.* § 23.

90. *A radius of a circle is a segment joining the center to a point on the circle.* § 24.

91. *An angle is a figure formed by two rays, or segments, having a common end point.* § 27.

92. *When one line meets another line so as to make two adjacent angles equal, each of these angles is called a right angle and the lines are said to be perpendicular to each other.* § 31.

93. *An angle is a straight angle if its sides lie in the same straight line and in opposite directions from the vertex.* § 32.

94. *Two angles are vertical if the sides of one are prolongations of the sides of the other.* § 38.

95. *Two angles are complementary if their sum is one right angle.* § 39.

96. *Two angles are supplementary if their sum is two right angles.* § 40.

97. *A triangle is a plane figure bounded by three line segments.* § 42.

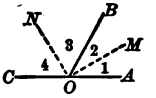
98. *Two geometric figures are equal if, without changing the shape or size of either, they can be made to coincide.* § 50.

99. *A proof is an argument which shows that a proposition follows from other known propositions.* § 68.

100. *A theorem is a proposition to be proved.* § 69.

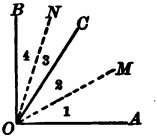
101. *An axiom is a proposition accepted without proof.* § 71.

EXERCISES ON ANGLES

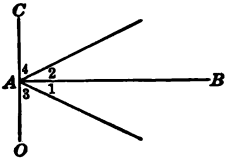


1. In the figure, $\angle AOC$ is a straight angle, OM bisects $\angle AOB$ and ON bisects $\angle BOC$. What kind of an angle is $\angle MON$? Why?

Suggestion: $\angle 2$ *is* $\frac{1}{2}\angle AOB$ and $\angle 3$ is $\frac{1}{2}$ of $\angle BOC$.



2. In the figure, $\angle AOB$ is a right angle, OM bisects $\angle AOC$ and ON bisects $\angle BOC$. How many degrees are there in $\angle MON$?



3. In the figure, $CO \perp AB$ and $\angle 1 = \angle 2$. What is the relation between $\angle 3$ and $\angle 4$?

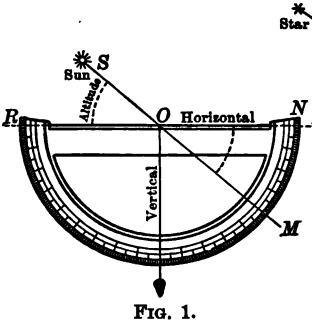


FIG. 1.

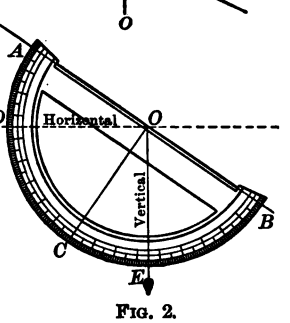


FIG. 2.

4. A protractor is fastened to a board and a plumb line is attached to it by means of a nail at the point O, as shown in figure 1. To get the altitude of the sun, hold the board vertically in line with the sun so that the line of the plumb bob falls on the 90° mark. The shadow of the nail at the center of the protractor falls across the scale showing the altitude of the sun. Explain this.

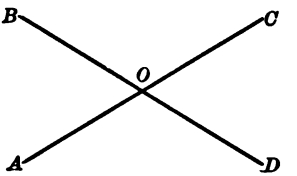
5. In figure 2, sight along the straight edge of the protractor at a star. Notice where the plumb line falls across the scale. Find the altitude of the star. Find your latitude by getting the altitude of the North Star.

BOOK I

RECTILINEAR FIGURES

VERTICAL ANGLES ARE EQUAL

102. THEOREM I. *If two lines intersect, the vertical angles are equal.*



Given lines AC and BD intersecting at O.

To prove that $\angle AOB = \angle COD$.

Proof: $\angle AOB + \angle BOC = 2$ rt. $\measuredangle$. § 81

(*Two adjacent* $\measuredangle$ *formed by one straight line meeting another straight line are together equal to two rt.* $\measuredangle$.)

$\angle BOC + \angle COD = 2$ rt. $\measuredangle$. § 81

$\therefore \angle AOB + \angle BOC = \angle BOC + \angle COD$. Ax. 5, § 71

(*Quantities which are equal to the same quantity or to equal quantities are equal to each other.*)

$\therefore \angle AOB = \angle COD$. Ax. 2, § 71

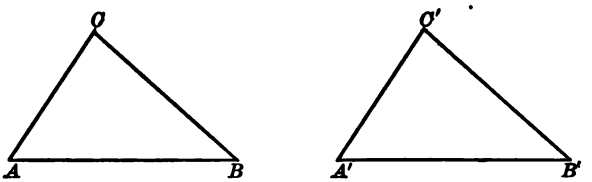
(*If equals are subtracted from equals, the remainders are equal.*)

Q. E. D.

103. Nature of Formal Proof. The above demonstration is given in *formal detail.* For an informal proof see § 79. In a formal proof a reason must be given for *each* step. The reason may be a definition, an axiom, or a theorem already proved.

EQUALITY OF TRIANGLES: FIRST TEST

104. THEOREM II. *Two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Given $\triangle ABC$ **and** $A'B'C'$ **in which** $AB = A'B'$, $AC = A'C'$, **and** $\angle A = \angle A'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof: Place $\triangle ABC$ on $\triangle A'B'C'$ so that $\angle A$ shall coincide with its equal $\angle A'$, AB falling along $A'B'$ and AC along $A'C'$. Ax. 17, § 71

(*A figure may be moved about without changing its shape or size.*)

Then B will fall on B' and C on C'.

(*For it is given that* $AB = A'B'$ *and* $AC = A'C'$.)

$\therefore$ line-segments BC and $B'C'$ coincide throughout. § 75

(*Two line-segments coincide if their end points coincide.*)

$\therefore \triangle ABC = \triangle A'B'C'$. § 98

(*Geometric figures are equal if they can be made to coincide.*)

Q. E. D.

105. COROLLARY. *Two right triangles are equal if the sides including the right angle in one are equal respectively to the sides including the right angle in the other.*

Suggestion. The right angles are equal by § 74.
Draw the triangles and show that the above theorem applies.

THE PARTS OF A THEOREM

106. Hypothesis. Conclusion. That part of a theorem which is given is called the *hypothesis*, while the part which is to be proved is called the *conclusion*.

Thus, in Theorem II, the hypothesis is: "if two sides and the included ∠ of one △ are equal respectively to two sides and the included ∠ of the other"; and the conclusion is: "the two ▲ are equal."

In giving a proof *all parts* of the hypothesis must be used. To make certain that this is done we may tabulate these parts and check off each one *as it is used*.

Thus, the hypothesis of Theorem II is: $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$. When, in the proof, $\angle A$ is placed on $\angle A'$, the equation $\angle A = \angle A'$ is checked off. When $AB = A'B'$ and $AC = A'C'$ are used, these equations are likewise checked off. This affords a valuable check on the work, since a proof which fails to use all the statements of the hypothesis is faulty.

SIGHT WORK

1. If, in figure 1, $OA = OA'$, $OB = OB'$, then $\triangle AOB = \triangle A'OB'$.

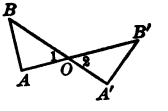


FIG. 1.

Suggestion: (1) $\angle 1 = \angle 2$. Why? Give the reason.

(2) Use Theorem II to show that $\triangle AOB = \triangle A'OB'$.

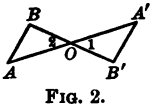


FIG. 2.

2. If, in figure 2, $OA = OA'$ and $OB = OB'$, then $\triangle AOB = \triangle A'OB'$. Give reasons in full.

3. If, in figure 3, $\angle 1 = \angle 2$ and $AB = AB'$, then $\triangle ABC = \triangle AB'C$. Give reasons in full.

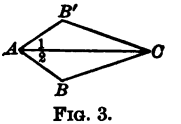


FIG. 3.

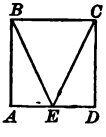


FIG. 4.

(A square is a four sided figure whose sides are all equal and whose angles are all right angles.)

4. If, in figure 4, $ABCD$ is a square and E is the middle point of AD, show that $\triangle AEB = \triangle DEC$.

Suggestion: (1) $\angle A = \angle D$. See § 74.

(2) Use corollary under Theorem II.

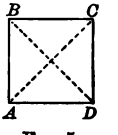
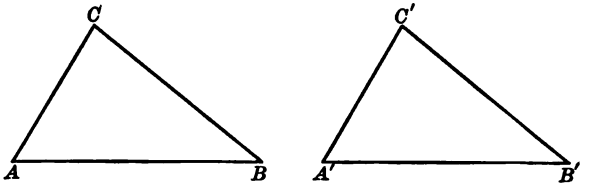


FIG. 5.

5. If $ABCD$ is a square, show that $\triangle ADB = \triangle ADC$.

EQUALITY OF TRIANGLES: SECOND TEST

107. THEOREM III. *Two triangles are equal if two angles and the included side of one are equal respectively to two angles and the included side of the other.*



Given $\triangle ABC$ **and** $A'B'C'$ **in which** $\angle A = \angle A'$, $\angle B = \angle B'$, **and** $AB = A'B'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof: Place $\triangle ABC$ on $\triangle A'B'C'$ so that AB shall coincide with its equal $A'B'$, making C fall on the same side of $A'B'$ as C'. Ax. 17, § 71

(*A figure may be moved about without changing its shape or size.*)

Then line AC will fall along $A'C'$ and BC along $B'C'$.

(*For it is given that* $\angle A = \angle A'$ *and* $\angle B = \angle B'$.)

$\therefore$ point C falls on both $A'C'$ and $B'C'$.

$\therefore$ C falls on C'. § 73

(*Two straight lines cannot have more than one point in common.*)

$\therefore$ $\triangle ABC = \triangle A'B'C'$. § 98

(*Two figures are equal if they can be made to coincide.*)

Q. E. D.

NOTE. — In the above theorem the hypothesis is "if two $\measuredangle$ and the included side of one $\triangle$ are equal respectively to two $\measuredangle$ and the included side of another"; and the conclusion is "the two $\triangle$ are equal."

In studying the proof notice just the step at which each part of the hypothesis is used.

SIGHT WORK

1. If, in figure 1, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, show that $\triangle ABC = \triangle ADC$.

Suggestion: AC is equal to itself. Use Theorem III.

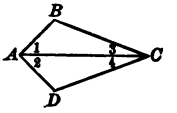


FIG. 1.

2. If, in figure 2, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, show that $\angle ABC = \angle ADC$. § 107.

3. If, in figure 2, $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$, show that $\triangle ABC = \triangle ADC$. § 107.

4. If, in figure 2, $\angle 1 = \angle 2$ and $AB = AD$, show that $\triangle ABC = \triangle ADC$. § 104.

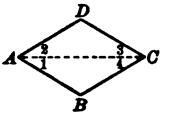


FIG. 2.

5. If, in figure 3, $\angle 3 = \angle 4$ and $BC = DC$, show that $\triangle ABC = \triangle ADC$. § 104.

6. If, in figure 3, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, show that $\triangle ABC = \triangle ADC$.

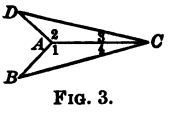


FIG. 3.

7. If, in figure 4, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, show that $\triangle ADB = \triangle CDB$.

8. If, in figure 4, $AB = BC$ and $\angle 3 = \angle 4$, show that $\triangle ADB = \triangle CDB$.

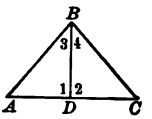


FIG. 4.

108. Designating Corresponding Parts of Figures. It sometimes avoids confusion to designate by special marks those parts of the figures which are given equal by the hypothesis.

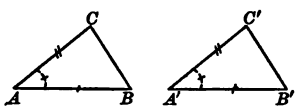


FIG. 1.

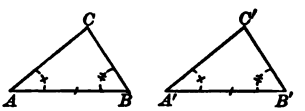


FIG. 2.

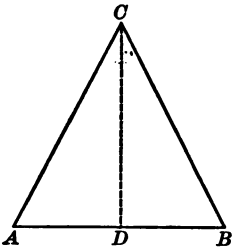
Thus, in Theorem II, it is given by hypothesis that $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$. The equal angles are indicated by marked arcs and the equal sides by single and double marks, respectively. See Fig. 1.

In Theorem III, it is given by hypothesis that $AB = A'B'$, $\angle A = \angle A'$, $\angle B = \angle B'$. This is indicated as shown in Fig. 2.

Colored crayons are sometimes used to designate corresponding parts, especially in complicated figures, but for ordinary purposes such marks as are suggested above are sufficient and are more convenient.

BASE ANGLES OF AN ISOSCELES TRIANGLE

109. THEOREM IV. *In an isosceles triangle the angles opposite the equal sides are equal.*



Given $\triangle ABC$ **in which sides** AC **and** BC **are equal.**

To prove that $\angle A = \angle B$.

Proof: Let CD be the bisector of $\angle C$.

Then in $\triangle ACD$ and BCD,

$\angle ACD = \angle BCD$, $AC = BC$, and $CD = CD$. § 41

(*For AC is given equal to BC and CD is identical to CD.*)

$\therefore \triangle ACD = \triangle BCD$. § 104

(*Two $\triangle$ are equal if two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.*)

$\angle A$ and $\angle B$ are corresponding angles of equal $\triangle$. § 52

$\therefore \angle A = \angle B$. § 76

(*Corresponding parts of equal figures are equal.*)

Q. E. D.

110. COROLLARY 1. *An equilateral triangle is equiangular.*

111. COROLLARY 2. *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.*

For, by the above proof, $\triangle ACD = \triangle BCD$.

$\therefore AD = DB$ and $\angle ADC = \angle BDC$. §§ 76, 92

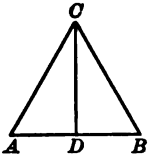
112. COROLLARY 3. *If the perpendicular from the vertex of a triangle to the base bisects the base, the triangle is isosceles.*

Proof: We have $\angle BDC = \angle ADC$, $CD = CD$, and $AD = DB$.

$\therefore \triangle ADC = \triangle BDC$. § 104

$\therefore AC = BC$. § 76

(*Corresponding parts of equal figures are equal.*)



SIGHT WORK

1. If $AB = AD$ and $BC = DC$, show that (*a*) $\angle 1 = \angle 2$, (*b*) $\angle 3 = \angle 4$, (*c*) $\angle ABC = \angle ADC$. Fig. 1.

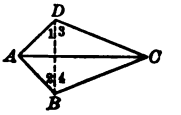


FIG. 1.

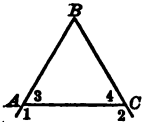


FIG. 2.

2. If $AB = BC$, show that $\angle 1 = \angle 2$. Fig. 2.

Suggestion: (*a*) $\angle 3 = \angle 4$, § 109; (*b*) $\angle 1 = \angle 2$, § 78.

3. Triangle ABC is equilateral. D, E, F are the middle points of its sides. Show that the $\triangle ADF = \triangle BDE$. Fig. 3.

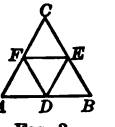


FIG. 3.

4. In figure 3, show that $\triangle DEF$ is equilateral.

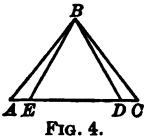


FIG. 4.

5. In $\triangle ABC$, $AB = BC$, and AE is laid off equal to CD. Show that $\triangle AEB = \triangle CDB$. Fig. 4.

6. In figure 5, $AE = ED = DC = CB$ and $\angle E = \angle C$. Show that $\triangle ADE = \triangle BDC$.

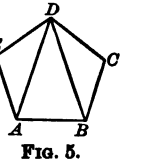


FIG. 5.

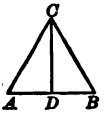


FIG. 6.

7. In the $\triangle ABC$, $AC = BC$ and $AD = BD$. Show that $\angle ADC = \angle BDC$. Fig. 6.

8. The four angles of the figure $ABCD$ are all right angles. Also $AD = BC$ and $DC = AB$. Show that $\triangle ABC = \triangle ACD$. Fig. 7.

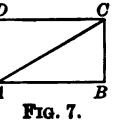
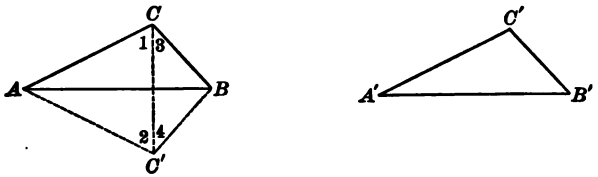


FIG. 7.

EQUALITY OF TRIANGLES: THIRD TEST

113. THEOREM V. *Two triangles are equal if three sides of one are equal respectively to three sides of the other.*



Given $\triangle ABC$ **and** $A'B'C'$ **in which** $AB = A'B'$, $BC = B'C'$, **and** $CA = C'A'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof: Place $\triangle A'B'C'$ so that $A'B'$ shall coincide with its equal AB and so that C' shall fall opposite C. Ax. 17, § 71

(*A figure may be moved about without changing its shape or size.*)

Draw CC'.

Then $\triangle ACC'$ and BCC' are isosceles. § 45

(*It is given that* $AC = AC'$ *and* $BC = BC'$.)

$\therefore \angle 1 = \angle 2$ and $\angle 3 = \angle 4$. § 109

(*In an isosceles* $\triangle$ *the* $\angle$ *opposite the equal sides are equal.*)

$\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$ or $\angle C = \angle C'$. Ax. 1, § 71

(*If equals are added to equals, the sums are equal.*)

Hence, in $\triangle ABC$ and $A'B'C'$, $AC = A'C'$, $BC = B'C'$, and $\angle C = \angle C'$.

$\therefore \triangle ABC = \triangle A'B'C'$. § 104

(*Two* $\triangle$ *are equal if two sides and the included* $\angle$ *of one, etc.*)

Q. E. D.

NOTE. — In the above proof the theorem of § 104 is quoted in part only. All theorems used in a proof should be stated in full by the pupil.

SIGHT WORK

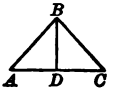


Fig. 1.

1. In figure 1, ABC is an isosceles triangle and $AD = DC$. Show that $\triangle ABD = \triangle CBD$.

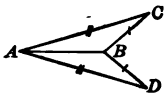


Fig. 2.

2. In figure 2, equal segments are indicated by marks according to § 108. Show that $\triangle ABC = \triangle ABD$.

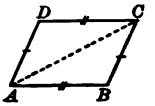


Fig. 3.

3. In figure 3, equal segments are indicated by marks. Show that $\triangle ABC = \triangle ADC$.

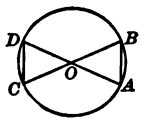


Fig. 4.

4. If O is the center of the circle, in figure 4, show that $\triangle AOB = \triangle COD$.

114. Determination of a Triangle. The theorems of §§ 104, 107, 113, giving the three tests for the equality of triangles, may be regarded as showing the different ways in which a triangle is *determined*. That is, a triangle is determined or fixed:

(1) if two sides and the included angle are given,

(2) if two angles and the included side are given,

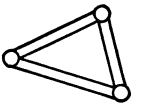
(3) if three sides are given.

This would seem to indicate that the remaining parts of a triangle may be computed when any of the three sets of parts just specified is given. This is, indeed, the case, but the actual computation is beyond the scope of elementary geometry and is deferred until the study of trigonometry is taken up.

115. Proving Equality of Angles and Segments by means of Equal Triangles. Since corresponding parts of equal figures are those which coincide when the figures are made to coincide, it is clear that two segments or two angles may be proved equal by showing that they are corresponding parts of equal figures.

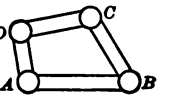
Thus, to prove that two segments are equal, we may show that they are corresponding sides of equal triangles. This method will now be used in solving a number of problems.

SIGHT WORK



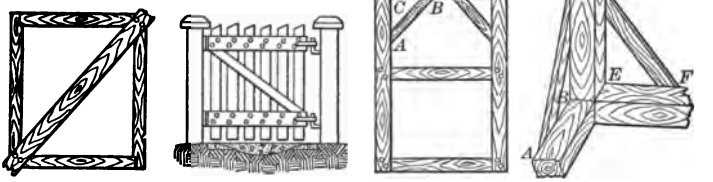
1. Three rods are hinged at the ends as shown in the figure. Is the figure rigid? Why? (See § 114.)

Remember that three sides of a triangle determine the triangle.



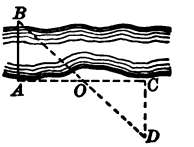
2. Four rods are hinged at the ends as shown in the figure. Is the figure rigid? Did you ever see a four-sided iron frame that could be folded up?

3. If still another rod is hinged across the figure from A to C in Exercise 2, will the figure become rigid? Use § 114.



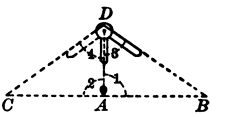
4. In each of the above show that the figure is rigid by using § 114.

5. To find the distance across a river from A to B, step off the distance from A to O along a line at right angles to AB. Continue in the same direction to C so that $AO = OC$. Then go along a line at right angles to OC until a point D in a line with O and B is reached. Show that $CD = AB$.



Suggestion: $\angle A = \angle C$. Why? $AO = OC$. $\angle BOA = \angle DOC$. Why? $\triangle BOA = \triangle DOC$. Why? Hence, $CD = AB$. Why?

6. To find the distance from A to an inaccessible point B, the following instrument is supposed to have been invented by Thales. (See Historical Note, page 2.) Two rods are linked together at D. One is placed vertical by means of a plumb line DA. The angle at D is so adjusted that the other rod points towards B. This arm is then swung around until it points toward an accessible point C. The distance AC is measured. Show that $AB = AC$.



EXERCISES

1. In figure 1, $AE = ED = DC = CB$ and $\angle E = \angle C$. Prove that $\angle 1 = \angle 2$.

Suggestion. Prove (1) $\triangle AED = \triangle BCD$; (2) $AD = BD$.

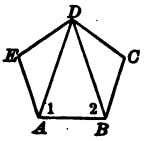


FIG. 1.

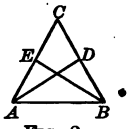


FIG. 2.

2. If $AC = BC$ and $AE = BD$, show that $BE = AD$. Fig. 2.

Suggestion: Show that $\triangle ADC = \triangle BEC$.

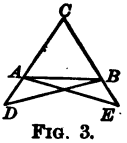


FIG. 3.

3. If $AC = BC$ and $AD = BE$, show that $BD = AE$. Fig. 3.

Suggestion: (1) $CD = CE$. (2) $\triangle CDB = \triangle CEA$. (3) $DB = EA$.

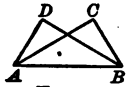


FIG. 4.

4. If $AD = BC$ and $\angle DAB = \angle CBA$, show that $AC = BD$. Fig. 4.

5. If $AD = AE$ and $BD = BE$, show that $DC = EC$. Fig. 5.

Suggestion: (*a*) $\triangle ABD = \triangle ABE$; (*b*) $\angle 1 = \angle 2$; (*c*) $\triangle BDC = \triangle BEC$.

6. If in example 5, it is given that $\angle 3 = \angle 4$ and $\angle ABD = \angle ABE$, prove $DC = EC$. Fig. 5.

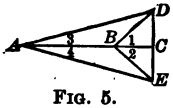


FIG. 5.

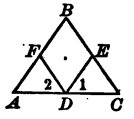


FIG. 6.

7. If $\angle A = \angle C$, $AD = CD$, and $\angle 1 = \angle 2$, show that $DE = DF$. Fig. 6.

8. If in the preceding example it is given that $AB = BC$, $AD = DC$, and $BF = BE$, show that $DE = DF$. Fig. 6.

9. To find the distance from A to B measure AO and BO. Then lay off $OD = AO$ and $OC = BO$. If BOC and AOD are straight lines, then $AB = CD$. Why? Hence, find AB by measuring CD. Fig. 7.

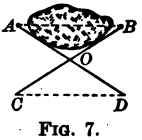


FIG. 7.

10. Prove the theorem of § 113 supposing $\angle B$ and $\angle B'$ to be obtuse instead of acute as in the figure on page 40.

GEOMETRIC CONSTRUCTION

116. Geometric Construction. We have already seen that the *construction* of geometric figures is to be performed with the ruler and the compasses only, and we have learned something about the uses of these instruments. We are now to study geometric construction more in detail, and in particular, we are to prove that the constructions which we make actually give the required figures.

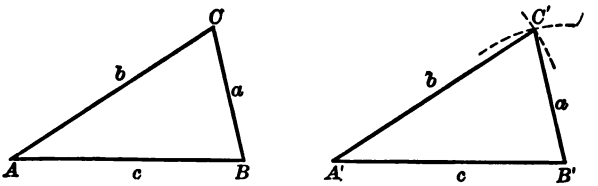
117. Steps in Solving a Construction Problem. In solving a construction problem the following steps are involved:

(1) A statement of what is given and what is required.
(2) Construction of the figure, with full description.
(3) Proof that the figure has the required properties.
(4) Discussion of the construction.

In many cases the properties of the figure are so obviously those required that little or no proof is needed.

CONSTRUCTION OF TRIANGLES

118. PROBLEM. *To construct a triangle equal to a given triangle.*



Given $\triangle ABC$ **with sides** a, b, c.

To construct $\triangle A'B'C'$ *equal to* $\triangle ABC$.

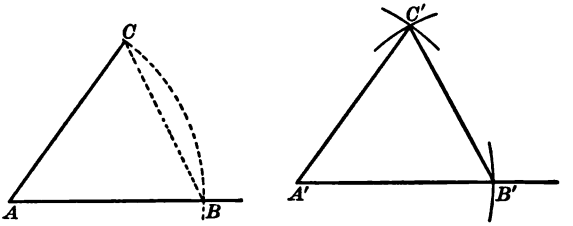
Construction: As in Example 2, page 20, construct triangle $A'B'C'$ with sides equal to a, b, c, respectively.

Then, $\triangle A'B'C' = \triangle ABC$. § 113

(*Two* ▲ *are equal if three sides of one are equal respectively to three sides of the other.*) Q. E. F.

GEOMETRIC CONSTRUCTION

119. PROBLEM. *To construct an angle equal to a given angle.*



Given angle A.

To construct $\angle A'$ *equal to* $\angle A$.

Construction: With A as center strike an arc meeting the sides of $\angle A$ at B and C. Draw BC.

As in § 118, construct $\triangle A'B'C' = \triangle ABC$.

Then $\angle A' = \angle A$. § 76

(*Corresponding parts of equal figures are equal.*) Q. E. F.

120. **Discussion of a Construction.** The discussion of a construction consists in pointing out the various essentially different results, which may arise under the given conditions, including those cases if any, in which the construction is impossible.

The following examples will show what is meant:

Example 1. Discuss the construction of § 118.

This construction is possible for any given triangle whatsoever.

Example 2. Discuss the construction of a triangle having sides equal to three given segments. (See page 20.)

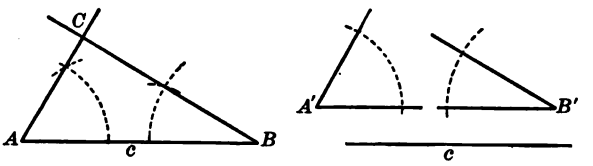
This construction is possible provided each of the given segments is less than the sum of the other two. The construction is impossible if one of the given segments is equal to or greater than the sum of the other two segments.

Example 3. Discuss the construction of § 119.

This construction is possible for any given angle whatsoever.

CONSTRUCTION OF TRIANGLES

121. PROBLEM. *To construct a triangle when two angles and the included side are given.*



Given $\angle A'$, $\angle B'$, and the line-segment c.

To construct a $\triangle$ having a side equal to c and the angles adjacent to it equal to $\angle A'$ and $\angle B'$ respectively.

Construction: On a line lay off segment $AB = c$.

Construct $\angle A = \angle A'$ and having AB as one of its sides. § 119

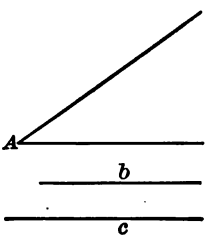
Construct $\angle B = \angle B'$ and having BA as one of its sides. § 119

Produce the remaining sides of $\angle A$ and $\angle B$ until they meet in a point C.

Then $\triangle ABC$ is the required triangle. Q. E. F.

Discussion. The discussion of this construction depends upon the sum of the angles of a triangle. See Example 4, page 67.

122. PROBLEM. *To construct a triangle when two sides and the included angle are given.*



Given $\angle A$ and line-segments b and c.

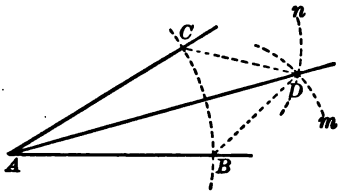
To construct a triangle having two sides equal to b and c respectively, and the included angle equal to $\angle A$.

Construction: The construction is left for the student.

Discussion. This construction is possible only when the given angle is less than a straight angle.

BISECTING AN ANGLE

123. PROBLEM. *To construct the bisector of an angle.*



To construct the bisector of a given angle A.

Construction: With A as a center and a convenient radius strike an arc meeting the sides of $\angle A$ in B and C.

With B and C as centers and with equal radii strike arcs m and n meeting at D. Draw AD.

Then AD is the required bisector of $\angle A$.

Proof: Draw BD and CD.

Then, $AB = AC$, $BD = CD$, $AD = AD$. § 85

$\therefore \triangle ABD = \triangle ACD$. § 113

(*Two triangles are equal if three sides of one, etc.*)

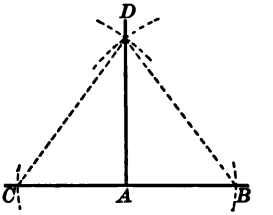
$\therefore \angle BAD = \angle CAD$, and AD bisects $\angle BAC$. §§ 76, 41

Q. E. F.

Discussion. This construction is possible for any angle.

124. PROBLEM. *To construct a perpendicular to a given line at a given point in it.*

Construction: Let A be the given point on the line BC. The problem amounts simply to the construction of the bisector of the straight angle A. The construction and the proof are exactly the same as in § 123.

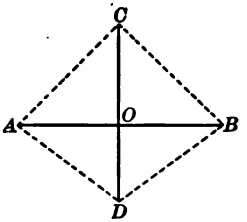


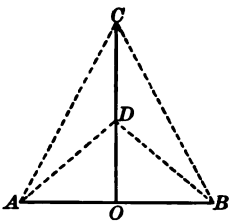
125. COROLLARY. *Not more than one perpendicular can be drawn to a line at a point in it.*

For $\angle 1$ and $\angle 2$ cannot both be right $\angle$s. Why?

PERPENDICULAR BISECTOR OF A LINE-SEGMENT

126. THEOREM VI. *Two points, each equidistant from the extremities of a line-segment, determine the perpendicular bisector of the segment.*





Given segment AB and points C, D such that $AC=BC$, $AD=BD$.

To prove that the line CD is the perpendicular bisector of AB.

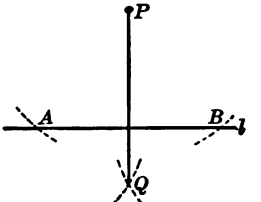
Proof: Draw the lines shown in the figures.

In either figure,	$AC=BC$, $AD=BD$, $CD=CD$.	Why?
	$\therefore \triangle ACD = BCD$.	§ 113
	$\therefore \angle ACD = \angle BCD$.	§ 76
Hence	$AC=BC$, $CO=CO$, $\angle ACO=\angle BCO$.	Why?
	$\therefore \triangle ACO = \triangle BCO$.	§ 104
	$\therefore AO=BO$ and $\angle AOC=\angle BOC$.	Why?
$\therefore$	CD is the perpendicular bisector of AB.	§ 92

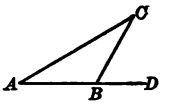
Q. E. D.

127. COROLLARY. *From a point not in a line to construct a perpendicular to the line.*

Suggestion. Let P and l be the given point and the given line. With P as a center and a fixed radius construct arcs meeting l in A and B. With A and B as centers and with equal radii construct arcs meeting in Q. Then PQ is the perpendicular bisector of AB by §126.



128. Exterior Angle of a Triangle. The angle formed by one side of a triangle and an adjacent side produced, is called an *exterior angle* of the triangle.



Thus, $\angle DBC$ is an exterior angle of the triangle ABC.

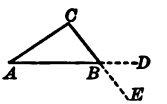
129. Adjacent and Opposite Interior Angles. The angle of the triangle which is adjacent to an exterior angle is the *adjacent interior angle* of the triangle, and the other two angles of the triangle are *opposite interior angles* of the triangle.

Thus, in the above figure $\angle ABC$ is the adjacent interior angle to $\angle DBC$, and $\angle A$ and $\angle C$ are the opposite interior angles to $\angle DBC$.

SIGHT WORK

1. How many exterior angles of a triangle can be formed at one vertex of a triangle? How are these angles related?

2. In the figure, is $\angle EBD$ an exterior angle of the triangle ABC? How are $\angle EBD$ and $\angle ABC$ related?



3. What is the sum of an exterior angle of a triangle and the adjacent interior angle?

EXPERIMENTAL GEOMETRY

In experimental exercises the figures constructed should be large and carefully drawn.

In each of the following draw several figures as different in shape as is permitted by the directions. Try to formulate your conclusion in each case as a definite statement or theorem.

1. Draw a triangle and produce one side to form an exterior angle. Measure the exterior angle and either of the two opposite interior angles. Which is larger? Test this by drawing triangles of different shapes.

2. Draw a triangle ABC such that $AB < BC$. Measure the angles opposite these sides. How do they compare?

3. Draw a triangle ABC such that $\angle A < \angle B$. Measure the sides opposite these angles. How do they compare?

4. Draw a triangle ABC such that $\angle A = \angle B$. Measure the sides opposite these angles. How do they compare?

EXTERIOR AND INTERIOR ANGLES OF A TRIANGLE

130. THEOREM VII. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

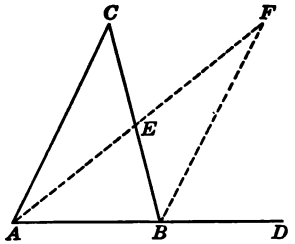


FIG. 1.

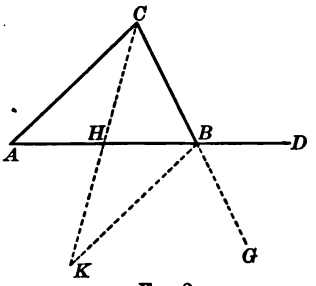


FIG. 2.

Given $\triangle ABC$ **with an exterior angle** CBD.

To prove that (1) $\angle CBD > \angle C$, (2) $\angle CBD > \angle A$.

Proof: (1) Let E be the middle point of BC. Fig. 1.

Draw AE and prolong it, making $EF = AE$. Draw BF.

Then $CE = EB$ and $AE = EF$. Why?

Also $\angle AEC = \angle BEF$. § 102

(*If two lines intersect, the vertical angles are equal.*)

$\therefore \triangle AEC = \triangle BEF$. § 104

$\therefore \angle C = \angle EBF$. § 76

But $\angle CBD > \angle EBF$. Ax. 7, § 71

(*The whole is greater than any of its parts.*)

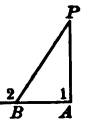
$\therefore \angle CBD > \angle C$. Ax. 9, § 71

(*Since* $\angle CBD > \angle EBF$ *and* $\angle EBF = \angle C$.)

(2) In Fig. 2, let $AH = HB$ and draw $HK = CH$. Then prove, as in Case (1), $\angle CBD = \angle ABG > \angle A$.

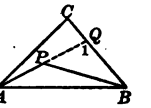
Q. E. D.

131. COROLLARY 1. *From a point not in a line one and only one perpendicular can be drawn to the line.*



Proof: In § 127 it was shown that it is always possible to construct one such perpendicular. We now show that there cannot be more than one. If, in the figure, both PA and PB are perpendicular to AB, then $\angle 1 = \angle 2$, both being right angles. But this is impossible, since by § 130 $\angle 2 > \angle 1$.

132. COROLLARY 2. *If P is any point within a triangle ABC, then $\angle APB > \angle C$.*



Proof: Produce AP to meet BC in Q. Then $\angle 1$ is an exterior angle of ΔACQ. Hence $\angle 1 > \angle C$. Why?

Again, $\angle APB$ is an exterior angle of ΔBQP. Hence $\angle APB > \angle 1$. Why? $\therefore \angle APB > \angle C$. Why? See Ax. 9, § 71.

NOTE. — It should be clear that corollaries are theorems which are distinguished from other theorems only in that they do not require *elaborate* proof, since they are easy consequences of the theorems under which they occur. But such proofs as are necessary should be given.

Problems in construction are sometimes placed as corollaries under theorems in case the method of construction and the proof are rendered obvious by the theorems, as in § 127.

SIGHT WORK

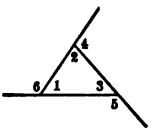


FIG. 1.

1. In the figure which is greater, $\angle 1$ or $\angle 4$, $\angle 3$ or $\angle 6$, $\angle 1$ or $\angle 5$, $\angle 3$ or $\angle 4$, $\angle 2$ or $\angle 6$, $\angle 2$ or $\angle 5$. Give reasons for answers.

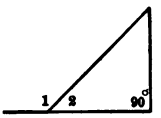


FIG. 2.

2. If one angle of a triangle is a right angle, show that the other two angles are acute.

Suggestion. Show that in the figure: (*a*) $\angle 1 >$ rt. $\angle$, (*b*) $\angle 2 <$ rt. $\angle$. See § 80.

3. In figure 3, compare $\angle 3$ and $\angle 2$ and also $\angle 1$ and $\angle 4$.

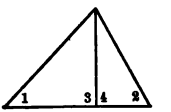


FIG. 3.

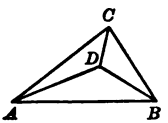


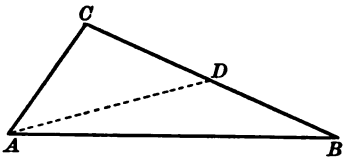
FIG. 4.

4. If any point D within a triangle is joined to the three vertices, show that the sum of the angles at D is greater than the sum of the angles of the triangle.

Suggestion. Use Corollary 2 above.

ANGLES OPPOSITE UNEQUAL SIDES ARE UNEQUAL

133. THEOREM VIII. *If two sides of a triangle are unequal, the angles opposite them are unequal, and the greater angle is opposite the greater side.*



Given $\triangle ABC$ **in which** $CB > CA$.

To prove that $\angle CAB > \angle B$.

Proof: On the segment CB lay off $CD = CA$ and draw AD.

Considering $\triangle ABD$ we have $\angle CDA > \angle B$. § 130

(*An exterior* $\angle$ *of a* $\triangle$ *is greater than either of the opposite interior* $\angle$*s.*)

But $\angle CAD = \angle CDA$. § 109

(*In an isosceles* $\triangle$ *angles opposite the equal sides are equal.*)

$\therefore \angle CAD > \angle B$. Ax. 14, § 71

(*A quantity may be substituted for its equal in an equation or in an inequality.*)

Again, $\angle CAB > \angle CAD$. Ax. 7, § 71

$\therefore \angle CAB > \angle B$. Ax. 9, § 71

(*Since* $\angle CAB > \angle CAD$ *and* $\angle CAD > \angle B$.)

Q. E. D.

134. **The Converse of a Proposition.** If the hypothesis and conclusion of one proposition are the conclusion and hypothesis, respectively, of another, then either is the converse of the other.

Thus, the theorem on page 53 is the converse of the one on this page.

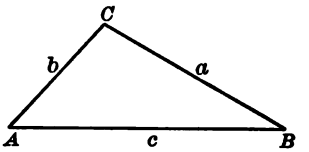
The converse of a true proposition is not always true.

Thus, "All right angles are equal" is a true proposition, but its converse, "All equal angles are right angles," is not true.

Hence, the converse of a proved theorem requires proof.

SIDES OPPOSITE UNEQUAL ANGLES ARE UNEQUAL

135. THEOREM IX. *If two angles of a triangle are unequal, the sides opposite them are unequal, and the greater side is opposite the greater angle.*



Given $\triangle ABC$ **in which** $\angle A > \angle B$.

To prove that $a > b$.

Proof: One of the following statements must be true:

(1) $a = b$, (2) $a < b$, (3) $a > b$. Ax. 8, § 71

(*If a, b, c, are quantities of the same kind, then* $a = b$, $a < b$ *or* $a > b$.)

But it cannot be true that $a = b$, for in that case $\angle A = \angle B$. § 109

(*In an isosceles* $\triangle$ *the angles opposite the equal sides are equal.*)

Again, it cannot be true that $a < b$, for in that case $\angle A < \angle B$. § 133

(*If two sides of a* $\triangle$ *are unequal, the* $\measuredangle$ *opposite them are unequal, and the greater* $\angle$ *is opposite the greater side.*)

$\therefore a > b$. Q. E. D.

136. **Proof by Exclusion.** The above argument is called *proof by exclusion.* It consists in showing (1) that at least one of certain enumerated cases must be true; and (2) that all but one of these cases are false. All but one of the cases are thus *excluded* and that one must be true.

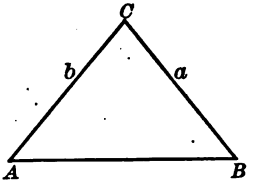
Thus, in the above proof we showed that,
(1) One of the cases $a = b$, $a < b$, $a > b$ must be true.
(2) The cases $a = b$ and $a < b$ are not true.
Therefore, $a > b$ is true.

SIDES OPPOSITE EQUAL ANGLES ARE EQUAL

137. THEOREM X. *If two angles of a triangle are equal, the sides opposite them are equal and the triangle is isosceles.*

Given $\triangle ABC$ **in which** $\angle A = \angle B$.

To prove that $a = b$.



Proof: One of the three following statements must be true:

(1) $a > b$, (2) $a < b$, (3) $a = b$.

Ax. 8, § 71

(*If a and b are quantities of the same kind, etc.*)

But it cannot be true that $a > b$, for in that case

$\angle A > \angle B$. § 133

(*If two sides of a* $\triangle$ *are unequal, the* $\angle$*s opposite them are unequal, and the greater angle is opposite the greater side.*)

For the same reason it cannot be true that $a < b$.

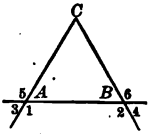
$\therefore a = b$. Q. E. D.

138. COROLLARY. *An equiangular triangle is also equilateral.*

SIGHT WORK

1. State the converse of the theorem of § 137. Has this already been given as a theorem?

2. If, in the figure, $\angle 1 = \angle 2$, prove that $\triangle ABC$ is isosceles. Also prove it isosceles if $\angle 3 = \angle 4$ or if $\angle 5 = \angle 6$.

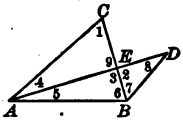


3. In the figure below, compare $\angle 3$ and $\angle 1$, $\angle 3$ and $\angle 4$, $\angle 9$ and $\angle 5$, $\angle 9$ and $\angle 6$, $\angle 2$ and $\angle 5$, $\angle 3$ and $\angle 8$.

4. In the same figure $AB > BD$. Compare $\angle 5$ and $\angle 8$.

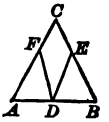
5. In the same figure $\angle 9 > \angle 4$. Compare AC and CE.

6. In the same figure $\angle 6 > \angle 1$. Compare AB and AC.



EXERCISES

1. In the triangle ABC, $AD = BD$, $DF = DE$, and $\angle ADF = \angle BDE$. Prove $\triangle ABC$ isosceles.

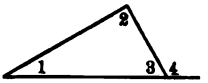


2. The sum of two angles of a triangle is less than two right $\measuredangle$s.

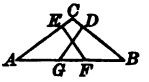
Suggestion. In the figure

$\angle 3 + \angle 4 = 2$ rt. $\measuredangle$ and $\angle 2 < \angle 4$.

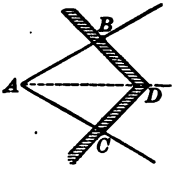
$\therefore \angle 3 + \angle 2 < 2$ rt. $\measuredangle$.



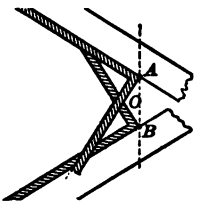
3. Construct an isosceles triangle ABC. At D and E equally distant from C draw perpendiculars to the equal sides, meeting the base at G and F, respectively. Prove $EF = DG$. See § 107.



4. A carpenter bisects an angle A as follows: Lay off $AB = AC$. Place a steel square so that $BD = CD$ as shown in the figure. Draw the line AD. Is this method correct? Give proof.

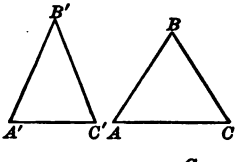


5. To cut two converging timbers by a line AB which shall make equal angles with them, a carpenter proceeds as follows: Place two squares against the timbers, as shown in the figure, so that $AO = BO$. Show that AB is the required line.



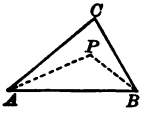
EXPERIMENTAL GEOMETRY

1. Draw two triangles ABC and $A'B'C'$ so that $AB = A'B'$, $BC = B'C'$ and $\angle B > \angle B'$. Compare the lengths of AC and $A'C'$ by measuring them.



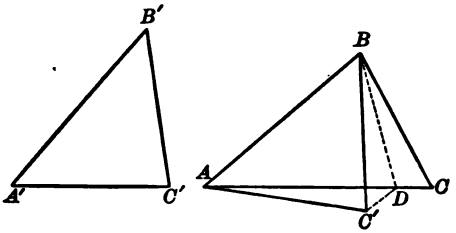
2. Construct triangles ABC and $A'B'C'$ so that $AB = A'B'$, $BC = B'C'$, and $AC > A'C'$. Compare $\measuredangle B$ and B' by measuring them.

3. Construct $\triangle ABC$ and from any point P within it draw segments PA and PB. Compare $AC + BC$ with $AP + BP$ by measurement.



UNEQUAL PARTS IN TWO TRIANGLES

139. THEOREM XI. *If in two triangles two sides of one are equal respectively to two sides of the other, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given $\triangle ABC$ **and** $A'B'C'$ **in which** $AB = A'B'$, $BC = B'C'$ **and** $\angle B > \angle B'$.

To prove that $AC > A'C'$.

Proof: Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'B'$ shall coincide with its equal AB and so that C' shall fall on the same side of AB as C.

Let BD bisect $\angle CBC'$, meeting AC in D. Draw $C'D$.

Then in $\triangle BCD$ and $BC'D$

$BC = BC'$, $\angle CBD = \angle C'BD$, and $BD = BD$. Why?

$\therefore \triangle BCD = \triangle BC'D$. § 104

$\therefore DC = DC'$. Why?

$\therefore AC = AD + DC = AD + DC'$. Ax. 14, § 71

(*A quantity may be substituted for its equal in an equation, etc.*)

But $AD + DC' > AC'$. Ax. 16, § 71

(*A line-segment joining two points is the shortest distance between them.*)

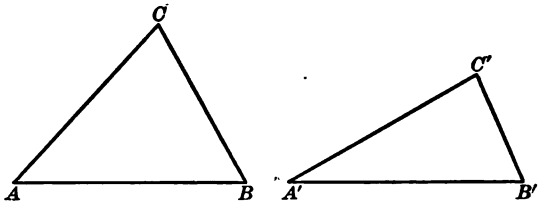
$\therefore AC > AC'$, that is, $AC > A'C'$. Ax. 10, § 71

(*Since* $AC = AD + DC' > AC'$.)

Q. E. D.

UNEQUAL PARTS IN TWO TRIANGLES

140. THEOREM XII. *If in two triangles two sides of one are equal respectively to two sides of the other, but the third side of the first is greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*



Given $\triangle ABC$ **and** $A'B'C'$ **in which** $AC = A'C'$, $AB = A'B'$, **and** $BC > B'C'$.

To prove that $\angle A > \angle A'$.

Proof: One of the following statements must be true:

(1) $\angle A = \angle A'$, (2) $\angle A < \angle A'$, (3) $\angle A > \angle A'$. Why?

But it cannot be true that $\angle A = \angle A'$; for in that case

$BC = B'C'$. Why?

Again, it cannot be true that $\angle A < \angle A'$, for in that case

$BC < B'C'$. § 139

$\therefore \angle A > \angle A'$. Why?

SIGHT WORK

1. What is the relation between Theorems XI and XII?

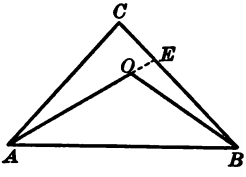
2. What kind of proof is used on this page?

3. If two $\triangle ABC$ and $A'B'C'$ are given such that $AB = A'B'$ and $BC = B'C'$, what must you know about $\angle B$ and $\angle B'$ to be able to conclude that $AC > A'C'$?

4. In the same $\triangle$ what must be known about AC and $A'C'$ to make sure that $\angle B > \angle B'$?

SEGMENTS FROM A POINT WITHIN A TRIANGLE

141. THEOREM XIII. *The sum of two sides of a triangle is greater than the sum of the segments drawn from any point within the triangle to the extremities of the third side.*



Given the $\triangle ABC$ and a point O within it.

To prove that $AC + BC > AO + BO$.

Proof: Extend AO to meet BC in the point E.

Then $AC + CE > AO + OE$. Ax. 16, § 71

(*A line-segment joining two points is the shortest distance between them.*)

and $OE + EB > BO$. Ax. 16, § 71

Adding these inequalities we have

$AC + CE + OE + EB > AO + OE + BO$. Ax. 12, § 71

(*If $a > b$ and $c > d$ then $a + c > b + d$.*)

Subtracting OE from both members,

$AC + CE + EB > AO + BO$. Ax. 11, § 71

(*If $a > b$, and $m = n$, then $a - m > b - n$.*)

But $CE + EB = BC$. Ax. 6, § 71

(*The whole is equal to the sum of its parts.*)

$\therefore AC + BC > AO + BO$. Ax. 14, § 71

(*A quantity may be substituted for its equal in an equation or, etc.*)

Q. E. D.

PARALLEL LINES

142. Parallel Lines. Two complete lines which lie in the same plane and which do not meet are said to be *parallel.*

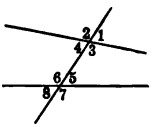
Two line-segments are parallel if they lie on parallel lines.

143. Axiom of Parallels. For the purpose of proving theorems XVI, XVII, and XVIII we need the following axiom.

AXIOM 21. *Through a point not on a given line only one straight line can be drawn parallel to that line.*

144. COROLLARY. *If each of two lines is parallel to a third line, they are parallel to each other.*

145. Angles Formed by Two Lines and a Transversal. A straight line which cuts two straight lines is called a *transversal* of these lines. The various angles formed by two lines and a transversal are named as follows:



$\angle 4$ and $\angle 5$ are *alternate-interior* angles; also $\angle 3$ and $\angle 6$.

$\angle 2$ and $\angle 7$ are *alternate-exterior* angles; also $\angle 1$ and $\angle 8$.

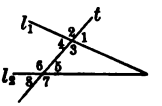
$\angle 1$ and $\angle 5$ are *corresponding* angles; also $\angle 3$ and $\angle 7$, $\angle 2$ and 6, $\angle 4$ and $\angle 8$.

$\angle 3$ and $\angle 5$ are *interior angles* on the same side of the transversal; also $\angle 4$ and $\angle 6$.

SIGHT WORK

1. In the above figure what pairs of angles are equal? What pairs of angles are supplementary?

2. If two lines meet and a transversal is drawn as in the figure, compare $\angle 1$ and $\angle 5$, $\angle 7$ and $\angle 3$, $\angle 6$ and $\angle 3$, $\angle 4$ and $\angle 5$.



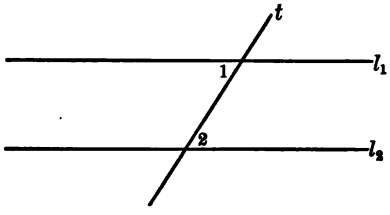
3. In the same figure is $\angle 3 + \angle 5$ greater or less than 2 rt. $\measuredangle$? See Exercise 2, page 55.

4. In the same figure is $\angle 4 + \angle 6$ greater or less than 2 rt. $\measuredangle$?

5. If in the figure $\angle 3$ and $\angle 5$ were both right angles, could the lines l_1 and l_2 meet? Why? See § 131.

EQUAL ALTERNATE INTERIOR ANGLES MAKE LINES PARALLEL

146. THEOREM XIV. *Two lines cut by a transversal are parallel if the alternate interior angles are equal.*



Given the lines l_1 and l_2 cut by the transversal t so as to make the equal alternate interior angles, $\angle 1$ and $\angle 2$.

To prove that $l_1 \parallel l_2$.

Proof: Suppose that l_1 and l_2 are not parallel. Then l_1 and l_2 will meet. Suppose they meet in a point to the right of t. Then a triangle is formed in which $\angle 1$ is an exterior angle and $\angle 2$ one of the opposite interior angles.

But this makes an exterior angle of a triangle equal to an opposite interior angle, which is impossible. § 130

(*An exterior $\angle$ of a $\triangle$ is greater than either of the opposite interior $\angle$s.*)

If l_1 and l_2 were to meet to the left of t, a triangle would be formed in which $\angle 2$ would be an exterior angle and $\angle 1$ an opposite interior angle. But this is impossible, for the same reason as before.

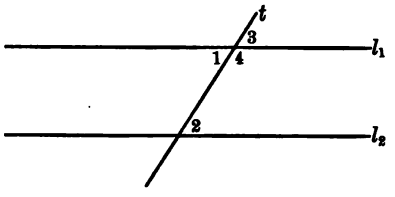
$\therefore$ l_1 and l_2 do not meet and are parallel. § 142

Q. E. D.

147. Indirect Proof. The type of proof used in § 146 is called *indirect proof.* It consists in showing that impossible results follow if the theorem is supposed not true. In the above instance we showed that if l_1 and l_2 were to meet, while at the same time $\angle 1 = \angle 2$, then the theorem of § 130 could not be true.

FURTHER CONDITIONS WHICH MAKE LINES PARALLEL

148. THEOREM XV. *Two lines cut by a transversal are parallel* (1) *if the corresponding angles are equal; or* (2) *if the sum of the interior angles on one side of the transversal is two right angles.*



Given transversal t cutting l_1 and l_2 forming $\angle 2$, $\angle 3$, and $\angle 4$, such that (1) $\angle 2 = \angle 3$, or such that (2) $\angle 2 + \angle 4 = 2$ rt. $\measuredangle$.

To prove that $l_1 \parallel l_2$.

Proof: (1) $\angle 3 = \angle 1$. § 102

(*If two lines intersect, the vertical angles are equal.*)

$\therefore \angle 1 = \angle 2$, Why?

and $l_1 \parallel l_2$. § 146

(2) By hypothesis $\angle 2 + \angle 4 = 2$ rt. $\measuredangle$.

But $\angle 1 + \angle 4 = 2$ rt. $\measuredangle$. § 81

$\therefore \angle 2 + \angle 4 = \angle 1 + \angle 4$. Ax. 5, § 71

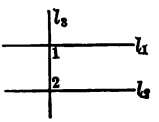
(*Quantities which are equal to the same quantity, etc.*)

$\therefore \angle 2 = \angle 1$. Ax. 2, § 71

$\therefore l_1 \parallel l_2$. § 146

Q. E. D.

149. COROLLARY. *Two lines which are perpendicular to the same line are parallel.*



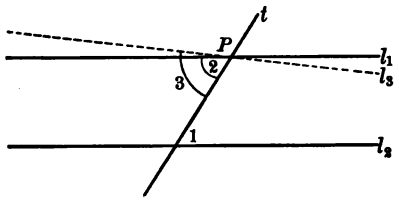
Proof. Since $\angle 1 =$ rt. $\angle$ and $\angle 2 =$ rt. $\angle$,

$\therefore \angle 1 + \angle 2 = 2$ rt. $\measuredangle$. Ax. 1, § 71

Hence, $l_1 \parallel l_2$. § 148

PARALLEL LINES MAKE ALTERNATE INTERIOR ANGLES EQUAL

150. THEOREM XVI. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Given $l_1 \parallel l_2$ and cut by the transversal t, forming the alternate interior angles, $\angle 1$ and $\angle 2$.

To prove that $\angle 1 = \angle 2$.

Proof: Suppose that $\angle 1$ is not equal to $\angle 2$.

Through P, the intersection of l_1 and t, draw l_3 so as to make $\angle 3 = \angle 1$.

Then, $l_3 \parallel l_2$. § 146

(*Two lines cut by a transversal are $\parallel$ if the alternate interior $\measuredangle$ are equal.*)

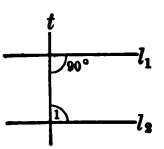
But by hypothesis, $l_1 \parallel l_2$ and we thus have two lines through the point P each parallel to a third line, which is impossible. Ax. 21, § 143

(*Through a point not on a given line only one straight line can be drawn parallel to that line.*)

Hence the supposition that $\angle 2$ is not equal to $\angle 1$ leads to a contradiction.

$\therefore \angle 2 = \angle 1$. Q. E. D.

151. COROLLARY 1. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

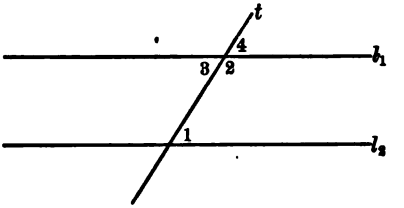


Suggestion. Prove that in the figure $\angle 1$ = rt. $\angle$.

152. COROLLARY 2. *If a line is parallel to one of two parallel lines, it is parallel to the other also.*

FURTHER PROPERTIES OF PARALLEL LINES

153. THEOREM XVII. *If two parallel lines are cut by a transversal, then* (1) *the corresponding angles are equal*; (2) *the sum of the interior angles on one side of the transversal is two right angles.*



Given $l_1 \parallel l_2$ and cut by the transversal t.

To prove (1) *that* $\angle 1 = \angle 4$ *and* (2) *that* $\angle 1 + \angle 2 = 2$ *rt.* $\measuredangle$.

Proof: (1) $\angle 1 = \angle 3$. § 150

But $\angle 4 = \angle 3$. § 102

(*If two lines intersect, etc.*)

$\therefore \angle 1 = \angle 4$. Ax. 5, § 71

(*Quantities which are equal to the same quantity, etc.*)

(2) $\angle 2 + \angle 3 = 2$ rt. $\measuredangle$. § 81

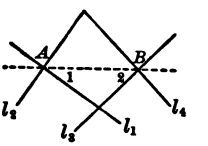
(*Two adjacent angles formed by one straight line meeting another, etc.*)

But $\angle 3 = \angle 1$. § 150

$\therefore \angle 2 + \angle 1 = 2$ rt. $\measuredangle$. Ax. 14, § 71

(*A quantity may be substituted, etc.*) Q. E. D.

154. COROLLARY. *If two lines are perpendicular to two intersecting lines respectively, then the given lines cannot be parallel.*



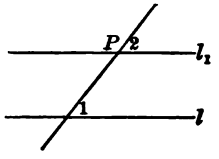
For let $l_1 \perp l_2$ and $l_3 \perp t_4$. Draw the transversal AB.

Show that $\angle 1 + \angle 2$ is less than 2 rt. $\measuredangle$.

NOTE. — It is interesting to observe that while Theorems XIV and XV can readily be proved without Axiom 21, § 143, it is impossible to prove Theorems XVI, XVII, and XVIII without this Axiom.

DRAWING ONE LINE PARALLEL TO ANOTHER

155. PROBLEM. *Through a given point to construct a line parallel to a given line.*



Given the line l and the point P outside it.

To construct a line l_1 through $P \parallel$ to l.

Construction: Through P draw a line making a convenient angle, $\angle 1$, with l.

Through P draw a line l_1 making $\angle 2 = \angle 1$. § 119

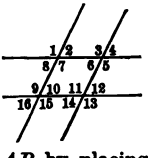
Then $l_1 \parallel l$. § 148

(*Two lines cut by a transversal are $\parallel$ if the corresponding $\measuredangle$ are equal.*)

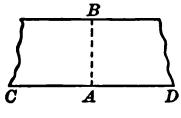
Q. E. F.

SIGHT WORK

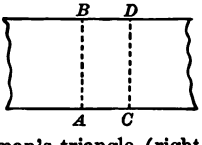
1. Two railway tracks cross as indicated in the figure. What angles are equal and what pairs of angles are supplementary?



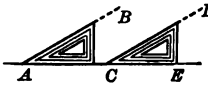
2. If $\angle 1 = 120°$, find $\angle 3$, $\angle 9$, $\angle 8$, $\angle 10$, $\angle 11$, $\angle 5$.



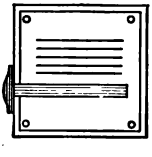
3. A paper is folded on the line AB by placing AD along AC (making $\angle BAD = \angle BAC$). Show that $\angle BAD$ is a right angle.



4. Show how to fold a sheet of paper so as to have the creases lie in parallel lines.



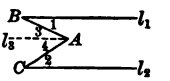
5. Show how a draughtsman's triangle (right triangle) can be used to draw parallel lines. (Show that AB and CD are parallel.)

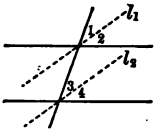


6. Show how the T-square can be used to draw parallel lines.

EXERCISES

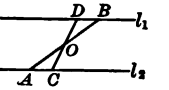
1. If in the figure $\angle BAC = \angle 1 + \angle 2$, then $l_1 \parallel l_2$.

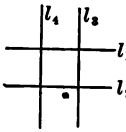




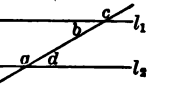
2. If parallel lines are cut by a transversal, then the bisectors of corresponding $\measuredangle$s are $\parallel$.

3. If in the figure $l_1 \parallel l_2$, and $AO = OB$, prove that $CO = OD$.

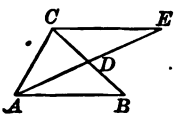




4. If in the figure l_1 and l_2 are parallel lines and $l_3 \perp l_2$ and $l_4 \perp l_1$, prove $l_3 \parallel l_4$.



5. If $l_1 \parallel l_2$ and $\angle a = 5 \times \angle b$, find $\angle c$ and $\angle d$.



6. Let ABC be any triangle. Bisect BC at D. Draw AD and prolong it to make $DE = AD$. Draw CE and prove $CE \parallel AB$.

Suggestion. Prove $\triangle CED = \triangle BAD$.

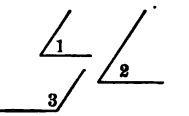
EXPERIMENTAL GEOMETRY

1. Construct a triangle ABC in which $\angle A = 60°$, $\angle B = 70°$; measure $\angle C$. What is the sum of the three angles?

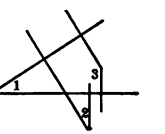
2. Construct a triangle ABC in which $\angle A = 110°$, $\angle B = 30°$. Measure $\angle C$. What is the sum of the three angles?

3. Draw several triangles, and measure the angles of each triangle. What seems to be the sum of the angles of a triangle?

4. Draw two angles so that the sides of one are parallel, respectively, to the sides of the other. Measure the angles. In the figure how are $\angle 1$ and $\angle 2$ related? $\angle 1$ and $\angle 3$?

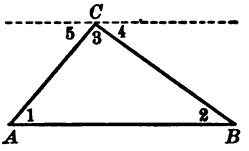


5. Draw two angles so that the sides of one are perpendicular, respectively, to the sides of the other. Measure the angles. In the figure how are $\angle 1$ and $\angle 2$ related? $\angle 1$ and $\angle 3$?



THE SUM OF THE ANGLES OF A TRIANGLE

156. THEOREM XVIII. *The sum of the angles of a triangle is two right angles.*



Given $\triangle ABC$ **whose angles are** $\angle 1$, $\angle 2$, **and** $\angle 3$.

To prove that $\angle 1 + \angle 2 + \angle 3 = 2$ rt. $\measuredangle$.

Proof: Through C draw a line $\parallel$ to AB forming $\angle 4$ and $\angle 5$.

Then $\angle 1 = \angle 5$ and $\angle 2 = \angle 4$. § 150

(*If two* $\parallel$ *lines are cut by a transversal, the alt. int.* $\measuredangle$ *are equal.*)

But $\angle 5 + \angle 3 + \angle 4 = 2$ rt. $\measuredangle$. § 83

(*The sum of all consecutive* $\measuredangle$ *about a point in a straight line and on one side of it is* 2 *rt.* $\measuredangle$.)

$\therefore \angle 1 + \angle 3 + \angle 2 = 2$ rt. $\measuredangle$. Ax. 14, § 71

(*A quantity may be substituted for, etc.*)

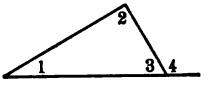
Q. E. D.

157. COROLLARY 1. *In a right triangle, there are two acute angles, and their sum is one right angle.*

158. COROLLARY 2. *If two angles of one triangle are equal respectively to two angles of another triangle, the remaining angles are equal.*

159. COROLLARY 3. *If an acute angle of one right triangle is equal to an acute angle of another right triangle, the remaining acute angles are equal.*

160. COROLLARY 4. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*



Suggestion. $\angle 1 + \angle 2 + \angle 3 = 2$ rt. $\measuredangle$, and $\angle 3 + \angle 4 = 2$ rt. $\measuredangle$.

Euclid (about 330–275 B.C.), a famous teacher of geometry who lived in Alexandria, Egypt, was the first to make a systematic presentation of all the investigations in geometry known in his time (see page 2). This book was called the *Elements* of Geometry.

One writer has said of this book: "Whatever has been claimed in praise of mathematics, of the strength, perspicuity, and rigor of its presentation, all is especially true of this work of the great Alexandrian."

It is said that, when asked if there were any shorter way to a knowledge of geometry than by his *Elements*, Euclid replied: "There is no royal road to geometry."

Euclid's "parallel axiom" has sufficed to distinguish this treatise as "Euclidean Geometry" in distinction from various other forms of later development.

HISTORICAL NOTES

The Axiom of Parallels. The so-called axiom of parallels has attracted more attention than any other proposition in geometry. Until the year 1829, persistent attempts were made by the world's most eminent mathematicians to prove it by means of the other axioms of geometry. In that year, however, a Russian, Lobachevsky, showed this to be impossible; and hence it must forever remain an axiom unless some other equivalent proposition is assumed.

The Sum of the Angles of a Triangle. The theorem that the sum of the angles of a triangle is two right angles is one of the most famous theorems of geometry. It was known to Pythagoras (550 B.C.) and special cases were known much earlier. The figure used in Exercise 1 below is the one given by Aristotle and Euclid. As is apparent, the proof depends upon the theorem, § 150, and thus indirectly upon Axiom 21. The interdependence of the propositions of §§ 150, 156 has been studied extensively during the last two centuries.

SIGHT WORK

1. In a right triangle one acute angle is 40°. What is the other angle?

2. If each of two angles of a triangle is 60°, what is the third angle?

In each of the following exercises ABC is a triangle. Find the number of degrees in $\angle C$.

3. $\angle A = 40°$, $\angle B = 80°$.

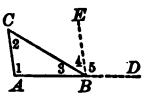
4. $\angle A = 70°$, $\angle B = 70°$.

5. $\angle A = 50°$, $\angle B = 50°$.

6. $\angle A = 45°$, $\angle B = 45°$.

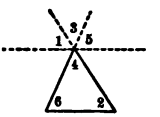
EXERCISES

1. Prove the theorem of § 156, using the figure in the margin.



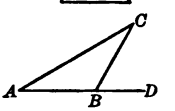
Suggestion. BE is parallel to AC. Then $\angle 4 = \angle 2$, and $\angle 5 = \angle 1$.

2. Prove the same theorem using the second figure in the margin.



Suggestion. Prove $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $\angle 5 = \angle 6$.

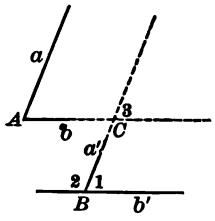
3. In the third figure $AB = BC$. Prove $\angle A = \frac{1}{2} \angle DBC$.



4. Use Theorem XVIII to discuss the construction in § 121.

ANGLES WHOSE SIDES ARE PARALLEL

161. THEOREM XIX. *Two angles whose sides are parallel, each to each, are either equal or supplementary.*



Given $\angle A$ with sides a and b and two lines a' and b' meeting at B and such that $a' \parallel a$ and $b' \parallel b$.

To prove that one of two adjacent $\measuredangle$s formed by a' and b' is equal to $\angle A$ and the other is supplementary to $\angle A$.

Proof: Extend b and a' to meet at a point C.

Then $\angle A = \angle 3$ and $\angle 3 = \angle 1$. § 153

(*If two $\parallel$ lines are cut by a transversal, the corresponding $\measuredangle$s are equal.*)

$\therefore \angle A = \angle 1$. Ax. 5, § 71

(*Quantities which are equal, etc.*)

But $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$s. Why?

$\therefore \angle A + \angle 2 = 2$ rt. $\measuredangle$s. Ax. 14, § 71

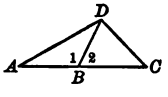
That is, $\angle A$ and $\angle 2$ are supplementary. § 96

Q. E. D.

SIGHT WORK

1. What is the sum of the angles of two triangles?

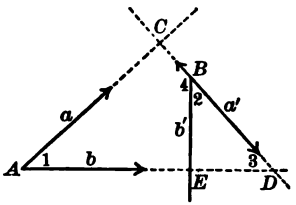
2. In the triangles ABD and BCD, $\angle 1$ and $\angle 2$ are supplementary. Is it possible that $\angle A$ and $\angle C$ are also supplementary? Why?



3. If in two triangles ABC and $A'B'C'$, $AB \parallel A'B'$, $BC \parallel B'C'$, and $CA \parallel C'A'$, how are the angles related? See §§ 158, 161.

ANGLES WHOSE SIDES ARE PERPENDICULAR

162. THEOREM XX. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



Given $\angle A$ with sides a and b, and two lines a' and b' meeting at B and such that $a' \perp a$ and $b' \perp b$.

To prove that one of two adjacent $\measuredangle$s formed by a' and b' is equal to $\angle A$ and the other is supplementary to $\angle A$.

Proof: Extend a and a' until they meet at C. Extend b until it meets b' and a' (extended if necessary) in E and D respectively.

Then $\triangle ACD$ and BED are rt. $\triangle$.

$\therefore \angle 1 + \angle 3 = 1$ rt. $\angle$, and $\angle 2 + \angle 3 = 1$ rt. $\angle$. § 157

(*The sum of the acute $\measuredangle$s of a rt. $\triangle$ is one rt. $\angle$.*)

$\therefore \angle 1 + \angle 3 = \angle 3 + \angle 2$. Why?

$\therefore \angle 1 = \angle 2$. Why?

But $\angle 2 + \angle 4 = 2$ rt. $\measuredangle$s. Why?

$\therefore \angle 1 + \angle 4 = 2$ rt. $\measuredangle$s. Why?

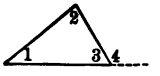
Q. E. D.

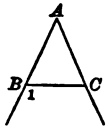
SIGHT WORK

If in two triangles ABC and $A'B'C'$, $AB \perp A'B'$, $BC \perp B'C'$, and $CA \perp C'A'$, how are the angles related?

EXERCISES

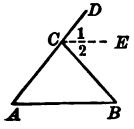
1. If in the figure $\angle 2 - \angle 1 = 15°$ and $\angle 4 = 120°$, find each angle of the triangle.



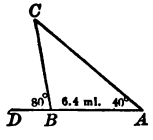


2. If in the figure $AB = AC$, find $\angle 1$ if $\angle A = 60°$; also, if $\angle A = 40°$. Show that, whatever the value of $\angle A$, $\angle 1 = 90° + \frac{1}{2}\angle A$.

3. Prove that the bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base.

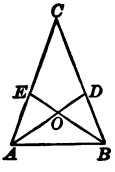


4. A sailor whose ship is at A and going in the direction AD notes that the angle DAC is 40°. After sailing 6.4 miles to B, he finds that $\angle DBC$ is 80°. He then concludes that the distance BC is 6.4 miles. Is this correct?

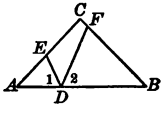


Suggestion. Prove $\triangle ABC$ isosceles. This is called "Doubling the angle at the bow."

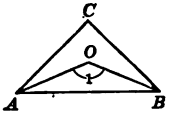
5. If the bisectors AD and BE of the base angles of an isosceles triangle ABC intersect in O, what pairs of equal angles are formed? What pairs of equal triangles are formed? of equal segments?



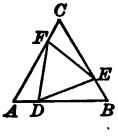
6. If in the isosceles $\triangle ABC$ a point D lies in the base and $\angle 1 = \angle 2$, determine whether there is any position for D such that $DE = DF$.



7. If in an isosceles right triangle ABC the bisectors of the acute angles meet at O, find how many degrees in the $\angle 1$ thus formed.

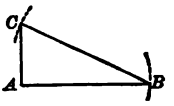


8. Triangle ABC is equilateral. $AD = BE = CF$. Compare the triangles DBE, ECF, FAD. Prove $\triangle DEF$ equilateral.



EXPERIMENTAL GEOMETRY

1. Construct a right triangle having given one side and the hypotenuse.



Solution: Construct a right angle A. On one side lay off AB equal to the given side. With B as center and the given hypotenuse as radius strike an arc cutting the other side of the right angle at C.

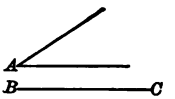
2. Construct two right triangles each with a side one inch and hypotenuse 2 inches. Compare the triangles by measuring sides and using § 113.

3. Construct two right triangles each with a side 2 inches and hypotenuse 3 inches. Compare triangles as in Example 2.

4. Construct two right triangles each with a side equal to a given segment and hypotenuse equal to another given segment. Compare triangles as in Example 2. What is your conclusion from Examples 2, 3, and 4?

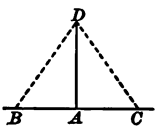
5. Construct two right triangles each with an acute angle of 60° and hypotenuse equal to 3 inches. Compare triangles as in Example 2.

6. Draw any acute angle A and a convenient segment BC. Construct two right triangles each having an angle equal to $\angle A$ and hypotenuse equal to BC. Compare the triangles by measuring sides and using § 113. What is your conclusion from Examples 5 and 6?



7. Construct AD perpendicular to BC at A. Lay off $AB = AC$ and draw BD and CD as shown in the figure. Measure and compare BD and CD.

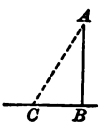
8. With $AD \perp BC$, as in Example 7, place one leg of the compasses at D and strike arcs meeting BC at B and C, thus making $BD = CD$. Measure and compare AB and AC.



9. Using lines BC and AD of Example 7, lay off $AC < AB$. Measure and compare CD and BD.

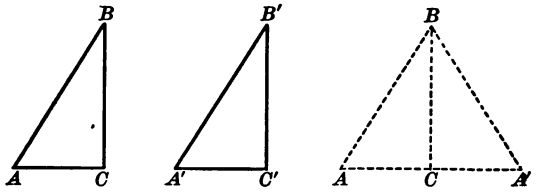
10. Using the same lines lay off $CD < BD$. Measure and compare AC and AB.

11. From a point A not on a line draw the perpendicular AB to the line. Also draw the oblique line AC. Measure and compare AB and AC.



EQUALITY OF RIGHT TRIANGLES

163. THEOREM XXI. *Two right triangles are equal if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*



Given the right $\triangle ABC$ **and** $A'B'C'$, **in which** $AB = A'B'$ **and** $BC = B'C'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof: Place the triangles so that BC shall coincide with its equal $B'C'$ and so that A and A' shall fall on opposite sides of BC.

Then ACA' is a straight line, because $\angle C$ and $\angle C'$ are rt. $\angle$s. § 82

(*If two adjacent* $\angle$s *are supplementary, their ext. sides lie in a straight line.*)

$\triangle ABA'$ is an isosceles triangle, and $\angle A = \angle A'$. Why?

$\therefore \angle B = \angle B'$. § 159

(*If an acute* $\angle$ *of one rt.* $\triangle$ *equals an acute* $\angle$ *of another rt.* $\triangle$ *the remaining acute* $\angle$s *are equal.*)

$\therefore \triangle ABC = \triangle A'B'C'$. § 107

(*Two* $\triangle$s *are equal if two* $\angle$s *and the included side of one are equal respectively to two* $\angle$s *and the included side of the other.*)

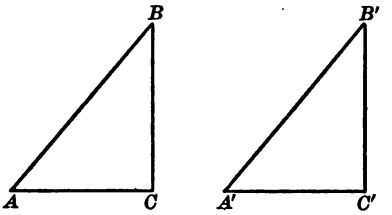
Q. E. D.

164. COROLLARY. *Two right triangles are equal if any two sides of one are equal respectively to the corresponding sides of the other.*

This follows from §§ 104 and 163.

EQUALITY OF RIGHT TRIANGLES

165. THEOREM XXII. *Two right triangles are equal if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.*



Given the right $\triangle ABC$ and $A'B'C'$ in which $AB = A'B'$ and $\angle A = \angle A'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof: Since $\angle A = \angle A'$ it follows that $\angle B = \angle B'$. § 159

(*If an acute ∠ of one rt. △ equals an acute ∠, etc.*)

Also it is given that $AB = A'B'$.

$\therefore \triangle ABC = \triangle A'B'C'$. § 107

(*Two ▲ are equal if two ∠ and the included side of one, etc.*)

Q. E. D.

166. COROLLARY. *Two right triangles are equal if any side and an acute angle of one are equal respectively to the corresponding side and acute angle of the other.*

State the various cases of this proposition.

SIGHT WORK

1. Two ladders of equal length have their lower ends resting on level ground at equal distances from a wall. Do they reach the same height on the wall? Why?

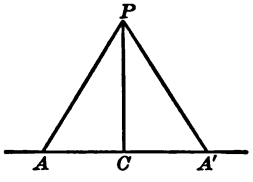
2. Is it possible that in two right triangles a leg and an acute angle of one may be equal to a leg and an acute angle of the other and the triangles not be equal? Draw figures to illustrate. Compare the exact wording of § 166.

EQUAL OBLIQUE SEGMENTS

167. THEOREM XXIII. *From a point in a perpendicular to a straight line oblique segments are drawn to the line. Then,*

(1) *if these segments are equal, they cut off equal distances from the foot of the perpendicular, and make equal angles with the perpendicular and with the given line; and*

(2) *conversely, if the distances cut off are equal, the segments are equal and make equal angles with the perpendicular and with the given line.*



Given $PC \perp AA'$ **and the oblique segments** PA **and** PA'.

(1) *If* $PA = PA'$, *we are to prove that* $CA = CA'$, $\angle A = \angle A'$ *and* $\angle APC = \angle A'PC$.

Proof: In the rt. $\triangle ACP$ and $A'CP$, $PA = PA'$ and $PC = PC$.

$\therefore \triangle APC = \triangle A'PC$. § 163

(*Two rt.* $\triangle$ *are equal if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*)

$\therefore CA = CA'$, $\angle A = \angle A'$, $\angle APC = \angle A'PC$. Why?

(2) *If* $CA = CA'$, *we are to prove that* $PA = PA'$, $\angle A = \angle A'$, *and* $\angle APC = \angle A'PC$.

Proof: In the rt. $\triangle APC$ and $A'PC$, $CA = CA'$ and $PC = PC$.

$\therefore \triangle APC = \triangle A'PC$. Why?

$\therefore PA = PA'$, $\angle A = \angle A'$ and $\angle APC = \angle A'PC$. Why?

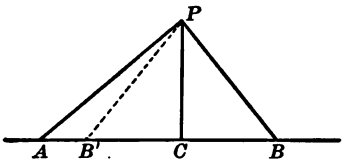
Q. E. D.

UNEQUAL OBLIQUE SEGMENTS

168. THEOREM XXXIV. *From a point in a perpendicular to a straight line oblique segments are drawn to the line. Then,*

(1) *if the distances cut off from the foot of the perpendicular are unequal, the oblique segments are unequal, that one being the greater which cuts off the greater distance; and*

(2) *conversely, if the oblique segments are unequal, the distances cut off are unequal, the greater segment cutting off the greater distance.*



Given $PC \perp AB$ and the oblique segments PA and PB.

(1) *If $CA > CB$, we are to prove that $PA > PB$.*

Proof: Let $CB' = CB$ and draw PB'.

Then	$\angle PAC <$ rt. $\angle$.	§ 157
Also	$\angle PB'C <$ rt. $\angle$ and $\therefore \angle PB'A >$ rt. $\angle$.	§ 80
	$\therefore PA > PB'$.	§ 135

(*If two $\measuredangle$ of a $\triangle$ are unequal, the sides opposite them are unequal and the greater side is opposite the greater angle.*)

But	$PB' = PB$.	§ 167, (2)
	$\therefore PA > PB$.	Ax. 9, § 71

(2) *If $PA > PB$, we are to prove that $CA > CB$.*

Proof: Show by (1) above that CA cannot be $< CB$.

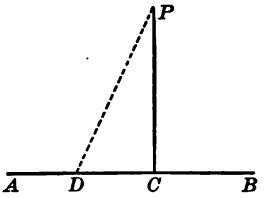
Then show by § 167 (2) that CA cannot $= CB$.

Hence $CA > CB$. Ax. 8, § 71

Q. E. D.

THE SHORTEST DISTANCE FROM A POINT TO A LINE

169. THEOREM XXV. *The perpendicular is the shortest segment from a point to a line.*



Given $PC \perp AB$ **and** PD **an oblique segment from** P **to** AB.

To prove that $PD > PC$.

Proof:

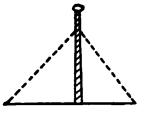
$\angle PCD > \angle PDC$.	§ 157
$\therefore PD > PC$.	§ 135

(*If two* $\measuredangle$ *of a* $\triangle$ *are unequal, etc.*) Q. E. D.

170. The **distance** from a point to a line means the *shortest distance* and hence is measured along the perpendicular.

SIGHT WORK

1. A vertical pole is braced by wires fastened near its top and in the ground. If two such wires are anchored in the ground at equal distances from the pole, what can one conclude as to the comparative lengths of the wires? Why? (It is assumed that the ground is level.)

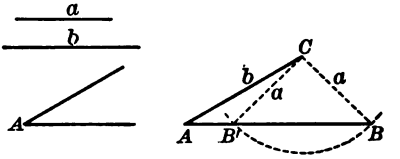


2. If in the preceding problem it is given that two such wires are anchored in the ground at unequal distances from the pole, what can be said about the length of the wires? Why?

3. If in the same problem we know that two such wires are unequal in length, what can we predict as to the distances from the pole at which they will have to be anchored in the ground?

CONSTRUCTION OF TRIANGLES

171. PROBLEM. *Given two sides of a triangle and an angle opposite one of them, to construct the triangle.*



Given $\angle A$ and the segments a and b.

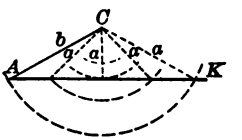
To construct a triangle having the sides a and b and the angle A opposite the side a.

Construction: On one side of $\angle A$ lay off $AC = b$, and let the other side be extended.

With C as a center and a radius equal to a, construct an arc of a circle as shown in the figure.

If, as in this case, a is given less than b, and greater than the perpendicular distance from C to the other side of the given angle, then there are two triangles having the given parts, namely ABC and $AB'C$.

Discussion. For different lengths of a the following cases are possible besides the case above.



(1) *a equal to the perpendicular from C to AK.* Then the arc will meet AK in but one point, and a right triangle is the only possible construction.

(2) *a less than the perpendicular.* Then the arc will not reach AK and the construction is impossible.

(3) $a = b$ or $a > b$. In either case only one construction is possible.

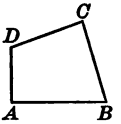
EXERCISES

1. Make the construction when $\angle A$ is a right angle.
2. Make the construction when $\angle A$ is obtuse.

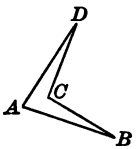
QUADRILATERALS

172. The Quadrilateral and its Parts. A plane figure bounded by four line segments is called a *quadrilateral.*

The segments are the *sides* of the quadrilateral and the points where they meet are its *vertices.*

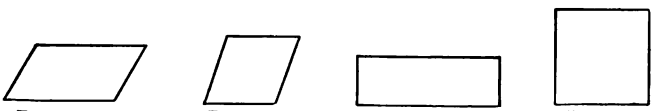


Two sides are *adjacent* if they meet in a vertex, as AB and BC. Otherwise they are *opposite*, as AB and CD. Two vertices are *adjacent* if they lie on the same side, as A and B. Otherwise they are *opposite*, as A and C.



A *diagonal* of a quadrilateral is a segment joining two opposite vertices.

Quadrilaterals which have a *reëntrant* angle, such as $\angle BCD$, are not considered here.



RHOMBOID RHOMBUS RECTANGLE SQUARE

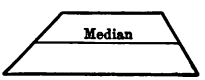
173. The Parallelogram. A *parallelogram* is a quadrilateral in which both pairs of opposite sides are parallel.

A *rhomboid* is a parallelogram whose angles are oblique. A *rhombus* is an *equilateral* rhomboid.

A *rectangle* is a parallelogram whose angles are right angles. A *square* is an equilateral rectangle.

The side on which a parallelogram is supposed to stand is called its *lower base*, and the side opposite is its *upper base.*

174. The Trapezoid. A *trapezoid* is a quadrilateral having only one pair of opposite sides parallel.

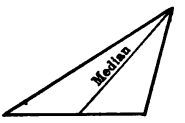


An *isosceles* trapezoid is one in which the two non-parallel sides are equal.

The two parallel sides of a trapezoid are its bases.

The *altitude* of a parallelogram or of a trapezoid is the perpendicular distance between its bases, and the *median* is the segment joining the middle points of the other sides.

175. Median of a Triangle. A segment connecting a vertex of a triangle with the middle point of the opposite side is called a *median of the triangle.*



SIGHT WORK

1. Are all squares rectangles? Is the converse statement true?

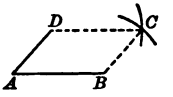
2. Are all squares parallelograms? Is the converse statement true?

3. Are all rectangles parallelograms? Is the converse statement true?

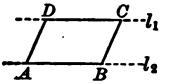
4. May a trapezoid have more than one pair of bases? May a parallelogram have more than one pair of bases?

EXPERIMENTAL GEOMETRY

1. As shown in the figure, construct a quadrilateral $ABCD$ so that $DC = AB$ and $BC = AD$. Test by § 148 (2) to see whether $BC \parallel AD$ and $DC \parallel AB$. That is, is the figure a parallelogram?

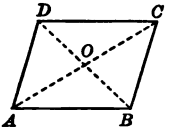


2. On two parallel lines l_1 and l_2, lay off equal segments AB and CD. Draw AD and BC as in the figure. Test to see whether $ABCD$ is a parallelogram.

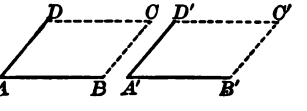


3. In Example 1, the opposite sides are constructed equal; in Example 2, two opposite sides are made equal and parallel. Does each of these conditions seem to be sufficient to make the figure a parallelogram?

4. Construct with care a large parallelogram $ABCD$. That is, make $AB \parallel DC$, and $BC \parallel AD$. Draw diagonals AC and BD meeting in O. Compare (1) AB and DC, AD and BC, (2) AO and OC, BO and OD, (3) $\triangle ABC$ and $\triangle ADC$. To compare the triangles use § 113.

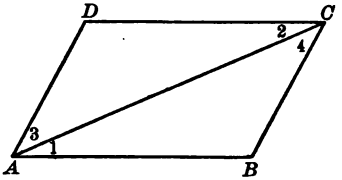


5. Construct parallelograms $ABCD$ and $A'B'C'D'$ so that $AB = A'B'$, $AD = A'D'$, and $\angle A = \angle A'$. Measure and compare BC and $B'C'$, CD and $C'D'$, $\angle B$ and $\angle B'$, $\angle C$ and $\angle C'$, $\angle D$ and $\angle D'$.



PROPERTIES MAKING A QUADRILATERAL A PARALLELOGRAM

176. THEOREM XXVI. *A quadrilateral is a parallelogram* (1) *if both pairs of opposite sides are equal; or* (2) *if two opposite sides are equal and parallel.*



(1) **Given a quadrilateral *ABCD* such that *AB* = *DC* and *BC* = *AD*.**

To prove that $ABCD$ *is a* $\square$.

Proof: Draw the diagonal AC forming the $\triangle ABC$ and CDA.

Then $AC = AC$, $AB = CD$, and $BC = AD$. Why?

$\therefore \triangle ABC = \triangle CDA$. § 113

$\therefore \angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Why?

$\therefore AB \parallel CD$ and $BC \parallel AD$. § 146

(*Two lines cut by a transversal are* ∥, *etc.*)

$\therefore ABCD$ is a $\square$. § 173

(*A* $\square$ *is a quadrilateral in which both pairs of opposite sides are parallel.*)

(2) **Given quadrilateral *ABCD* such that *AB* ∥ *CD* and *AB* = *CD*.**

To prove that $ABCD$ *is a* $\square$.

Proof: In $\triangle ABC$ and CDA, $AB = CD$, $AC = AC$. Why?

Also $\angle 1 = \angle 2$. § 150

(*If two* ∥ *lines are cut by a transversal, the alt. int.* ∠*s are equal.*)

$\therefore \triangle ABC = \triangle CDA$. Why?

$\therefore \angle 3 = \angle 4$ and $AD \parallel BC$. Why?

$\therefore ABCD$ is a $\square$. § 173

Q. E. D.

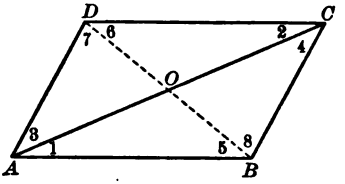
PROPERTIES OF A PARALLELOGRAM

177. THEOREM XXVII. *In a parallelogram*

(1) *a diagonal divides it into two equal triangles.*

(2) *the two diagonals bisect each other.*

(3) *opposite sides and opposite angles are equal.*



Given a parallelogram *ABCD* with diagonals *AC* and *BD*.

(1) *To prove that* $\triangle ABC = \triangle ADC$.

Proof: $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. § 150

$\therefore \triangle ABC = \triangle ADC$. § 107

(2) *To prove that* $AO = OC$ *and* $BO = OD$.

Proof: In the $\triangle ABO$ and CDO, $AB = CD$. Why?

Also $\angle 5 = \angle 6$ and $\angle 1 = \angle 2$. Why?

$\therefore \triangle ABO = \triangle CDO$. § 107

$\therefore AO = OC$ and $BO = OD$. Why?

(3) *To prove that* $AB = CD$ *and* $BC = AD$;

also that $\angle A = \angle C$ *and* $\angle B = \angle D$.

Proof: In the equal $\triangle ABO$ and CDO, $AB = CD$.

Similarly $\triangle AOD = \triangle BOC$ and hence $AD = BC$.

Again $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Why?

$\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$, or $\angle A = \angle C$. Ax. 1, § 71

In the same manner we may show that $\angle B = \angle D$. Q. E. D.

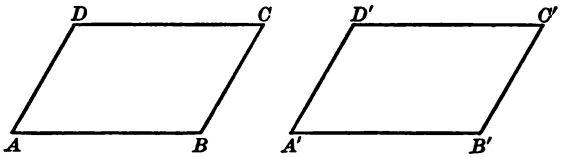
178. COROLLARY 1. *If two parallel lines are cut by two parallel transversals, two pairs of equal segments are intercepted.*

179. COROLLARY 2. *Parallel lines are everywhere equally distant.*

Suggestion. Use transversals that are $\perp$ to the two given $\parallel$s.

EQUAL PARALLELOGRAMS

180. THEOREM XXVIII. *Two parallelograms are equal if an angle and the two adjacent sides of one are equal respectively to an angle and the two adjacent sides of the other.*



Given $\boxed{S}$ $ABCD$ **and** $A'B'C'D'$ **in which** $AB = A'B'$, $AD = A'D'$, **and** $\angle A = \angle A'$.

To prove that $\square ABCD = \square A'B'C'D'$.

Proof: Apply $\square ABCD$ to $\square A'B'C'D'$ making $\angle A$ coincide with its equal $\angle A'$ and making AB fall along $A'B'$, and AD along $A'D'$.

Then B will fall on B' and D on D'.

(*It is given that* $AB = A'B'$ *and* $AD = A'D'$.)

Now $\angle B$ and $\angle B'$ are the supplements of $\angle A$ and $\angle A'$ respectively. § 153 (2)

(*If two* ∥ *lines are cut by a transversal, the sum, etc.*)

$\therefore \angle B = \angle B'$. § 78

(*∡s which are supplements of the same ∠ or of equal ∡s are equal.*)

$\therefore$ BC takes the direction $B'C'$.

$BC = B'C'$ because $BC = AD = A'D' = B'C'$. § 177, (3)

(*In a* $\square$ *opposite sides are equal.*)

$\therefore$ C falls on C'.

Hence CD coincides with $C'D'$. § 75

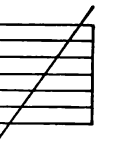
(*Two line-segments coincide if their end-points coincide.*)

Therefore $\square ABCD = \square A'B'C'D'$. § 98

(*Two figures are equal if they can be made to coincide.*) Q. E. D.

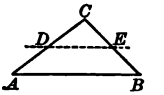
EXPERIMENTAL GEOMETRY

1. Take a sheet of ruled paper and draw a line across it. How is this line divided by the ruling lines?



2. Draw ‖ lines 1 in. apart. Draw any two transversals to these lines. Compare the lengths of the segments cut off on each transversal by the lines. Repeat, using another transversal. State a proposition which seems to be verified by Examples 1 and 2.

3. Construct a triangle ABC and through D, the middle point of AC, draw DE parallel to AB, meeting BC in E. Measure BE and EC, and compare them. State a proposition which this seems to justify.



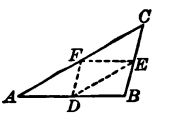
4. Draw a trapezoid and through the middle point of one side draw a line parallel to the bases. Does this line bisect the other side of the trapezoid? State a proposition which this experiment seems to verify.

5. If, in Example 3, DE is drawn connecting the middle points of AC and BC, then test to find whether DE is parallel to the base of the triangle. State a proposition which this seems to justify.

6. Draw a triangle ABC and connect the middle points D and E of AC and BC. Compare the length of the segments DE and AB. Repeat this experiment with several triangles and state a proposition suggested by your experiments.

7. Construct a trapezoid with bases 6 in. and 4 in. Draw the median, measure it, and compare it to the sum of the bases. Repeat this construction, using bases 5 in. and 3 in. State a proposition which this experiment seems to verify.

8. Draw a large scalene triangle and connect the middle points of its sides. Compare the resulting four triangles by cutting them out and superposing them.



SIGHT WORK

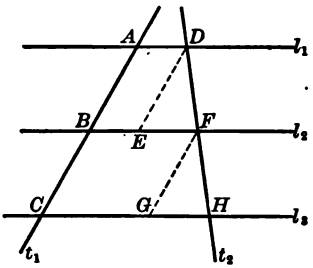
1. Define a parallelogram. What must be known about a quadrilateral to make it a parallelogram?

2. Name several properties of a parallelogram which follow from the definition.

3. What kind of proof is used in the theorem of § 180? What theorems about triangles were proved in similar manner?

EQUAL SEGMENTS INCLUDED BETWEEN PARALLELS

181. THEOREM XXIX. *If parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given $l_1 \parallel l_2 \parallel l_3$ **and cutting the transversal** t_1, **so that** $AB = BC$.

To prove that on the transversal t_2, $DF = FH$.

Proof: Draw DE and FG each parallel to t_1.

Then $\angle ABE = \angle BCG = \angle DEF = \angle FGH$. § 153 (1)

(*If two* ∥ *lines are cut by a transversal, the corresponding* ∡ *are equal.*)

Also $\angle EDF = \angle GFH$. (Since $DE \parallel FG$. § 144)

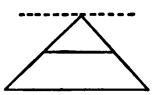
Again, $DE = AB = BC = FG$. § 178

(*If two* ∥ *lines are cut by two* ∥ *transversals two pairs of equal segments are intercepted.*)

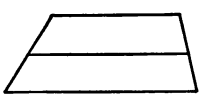
$\therefore \triangle DEF = \triangle FGH$ and $DF = FH$. § 107

Q. E. D.

182. COROLLARY 1. *If a line parallel to the base of a triangle bisects one side, then it bisects the other also.*

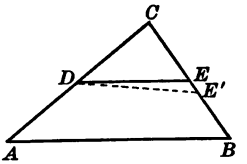


183. COROLLARY 2. *If a line parallel to the bases of a trapezoid bisects one side, then it bisects the other also.*



PROPERTIES OF TRIANGLES AND TRAPEZOIDS

184. COROLLARY 3. *A segment connecting the middle points of two sides of a triangle is parallel to the third side.*

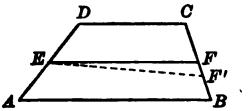


Proof: Let DE connect the middle points of CA and CB. Suppose DE not parallel to AB. Draw DE' parallel to AB.

Then $CE' = E'B$. § 182

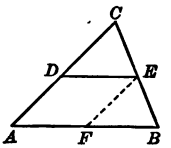
But $CE = EB$ and hence E and E' coincide, and DE is parallel to AB.

185. COROLLARY 4. *The segment connecting the middle points of the two non-parallel sides of a trapezoid is parallel to the bases.*



Proof: The proof is identical with that given in § 184 using § 183 instead of § 182. Give the proof in full.

186. COROLLARY 5. *A segment connecting the middle points of two sides of a triangle is equal to one-half the third side.*



Proof: Let DE connect the middle points of AC and BC, and draw $EF \parallel DA$. Then by § 184 $DE \parallel AB$.

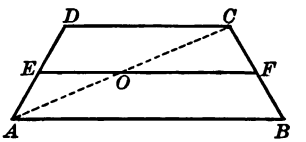
In $\triangle BFE$ and EDC, $\angle B = \angle DEC$ and $\angle FEB = \angle C$.

Also $BE = EC$ by hypothesis.

$\therefore \triangle BFE = \triangle EDC$ and $FB = DE$.

Since $DE = AF$ (§ 177) and $DE = FB$, then $DE = \frac{1}{2} AB$.

187. COROLLARY 6. *A segment connecting the middle points of the non-parallel sides of a trapezoid is equal to half the sum of the bases.*



Proof: Draw the diagonal AC meeting the median in O.

Then $EF \parallel AB$ and $EF \parallel DC$. § 185

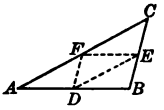
$\therefore O$ is the middle point of AC. § 182

$\therefore EO = \frac{1}{2} DC$ and $OF = \frac{1}{2} AB$. § 186

$\therefore EF = \frac{1}{2} (AB + DC)$.

SIGHT WORK

1. In the triangle ABC, lines joining the middle points of the sides are drawn as shown in the figure. If $AB = 22$ units, $BC = 14$, and $AC = 28$ units, find DE, DF, and FE.

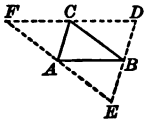


2. Through the vertices of the triangle ABC, lines are drawn parallel to the opposite sides as shown in the figure.

(a) What kind of a figure is $ABDC$?

(b) Compare AB and CD; also BD and AC.

(c) In the same manner compare AB and CF, BC and AF, AC and BE, CB and AE.



3. In the figure of Example 2,

(a) Is BC equal to one half EF? If so, why?

(b) In the same manner compare AC and ED, AB and FD.

4. If, in the figure of the preceding examples, $AB = 3$ in., $BC = 3\frac{1}{2}$ in., and $AC = 2\frac{1}{2}$ in., find DF, FE, and ED.

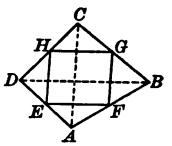
5. How may we test whether or not several parallel lines are equidistant by using an ordinary ruler? Is it necessary that it should be placed at right angles to the lines?

6. If the bases of a trapezoid are 5 in. and 2 in., what is the length of the median?

7. If one base of a trapezoid is 6 in. and the median is 5 in., what is the other base?

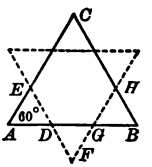
EXERCISES

1. The middle points of the sides of a quadrilateral are connected as shown in the figure. Prove that these connecting lines form a parallelogram.



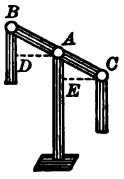
Suggestion. Prove EF and GH each $\parallel DB$ and hence to each other. Similarly prove $EH \parallel FG$.

2. Each side of an equilateral triangle is divided into three equal parts and lines are drawn joining them as shown in the figure. Prove that $\angle A = \angle F = \angle B$, etc.

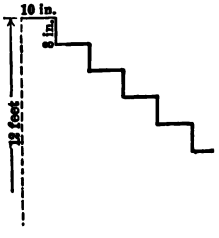


Suggestion. $\angle A = 60°$, $AD = AE$, $\therefore \angle EDA = \angle FDG = 60°$. In the same manner $\angle DGF = 60°$, $\therefore \angle F = 60°$.

3. Bars pivoted so as to move freely are attached to a standard as shown in the figure. If $AB = AC$, show that the perpendicular distances AD and CE are equal, no matter how much BC is tipped.

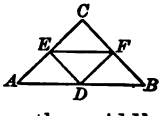


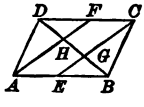
4. The steps leading from a floor to one 12 feet above it are 8 inches high and 10 inches wide. What is the length of the carpet required to cover the stairway, allowing 10 inches for the last step, which is on a level with the upper floor?



5. A stairway inclined 45° to the horizontal leads to a floor 15 feet above the first. What is the length of the carpet required to cover it if each step is 10 inches high? 9 inches?

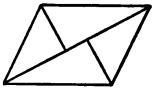
6. ABC is an isosceles $\triangle$. D is the middle point of the base, $DE \parallel BC$ and $DF \parallel AC$. Draw EF and prove the four triangles equal.

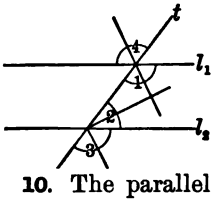




7. In $\square ABCD$, E and F are the middle points of AB and CD respectively. Show that AF and CE divide BD into three equal segments.

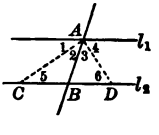
8. In the parallelogram perpendiculars are drawn from opposite vertices to diagonal. Prove that these perpendiculars are equal.





9. If $l_1 \parallel l_2$, how are the bisectors of $\angle 1$ and $\angle 3$ related? Of $\angle 3$ and $\angle 4$? Of $\angle 1$ and $\angle 2$? Use § 156 for the last case.

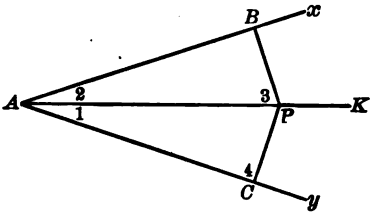
10. The parallel lines l_1 and l_2 are cut by a transversal AB. AC and AD bisect the interior angles at A respectively. Prove $\triangle CBA$ and DBA isosceles. Compare CB and BD.



POINTS EQUIDISTANT FROM THE SIDES OF AN ANGLE

188. THEOREM XXX. (1) *Every point on the bisector of an angle is equidistant from the sides of the angle; and*

(2) *conversely, if a point is equidistant from the sides of an angle it lies on the bisector of the angle.*



(1) **Given** AK **bisecting** $\angle A$, **and** P **a point on** AK.

To prove that P *is equidistant from* Ax *and* Ay.

Proof: Draw PB and PC perpendicular to Ax and Ay respectively.

In the rt. $\triangle$ ABP and ACP, we have

$$\angle 1 = \angle 2 \text{ and } AP = AP.$$

$$\therefore \triangle ABP = \triangle ACP. \qquad \S\ 165$$

(*Two rt.* $\triangle$ *are equal if the hypotenuse and an acute* $\angle$ *of one are equal respectively to the hypotenuse and an acute* $\angle$ *of the other.*)

$$\therefore PB = PC. \qquad \text{Why?}$$

(2) **Given** $PB \perp Ax$, $PC \perp Ay$ **and** $PB = PC$.

To prove that AP *bisects the given angle.*

Proof:

$$PB = PC \text{ and } AP = AP.$$

$$\therefore \triangle ABP = \triangle ACP. \qquad \S\ 163$$

(*Two rt.* $\triangle$ *are equal if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*)

$$\therefore \angle 1 = \angle 2. \qquad \text{Why?}$$

Q. E. D.

LOCI

189. From § 188 we are enabled to assert the following:

(1) *Every point in the bisector of an angle is equidistant from its sides.*

(2) *Every point equidistant from the sides of an angle lies in its bisector.*

For these reasons the bisector of an angle is called the **locus** of all points equidistant from its sides.

The word *locus* means *place* or *position*. It gives the location of all points having a given property.

190. All points in a plane which possess some specified property, as in the case preceding, will in general be restricted to a certain geometric figure.

A figure is called the **locus of all points** possessing a given property, provided:

(1) *Every point in the figure possesses the given property.*

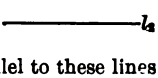
(2) *Every point in the plane which possesses the given property lies in the figure.*

If a locus is regarded as the path of a moving point, we may say:

A locus is a figure traced by a point moving in accordance with specified conditions.

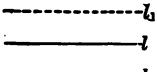
SIGHT WORK

1. The lines l_1 and l_2 are each parallel to l and at a distance of $\frac{1}{4}$ inch from it. What is the locus of all points in the plane $\frac{1}{4}$ inch distant from l?



2. What is the locus of all points in a plane at a given distance from a given line?

3. The lines l_1 and l_2 are parallel. The line l is parallel to these lines and midway between them. What is the locus of all points in the plane equidistant from l_1 and l_2?

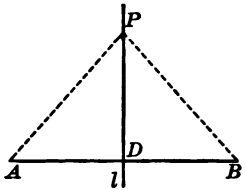


4. What is the locus of all points in a plane equidistant from two parallel lines in it?

5. What is the locus of all points in a plane 3 inches distant from a fixed point in the plane? Draw this locus.

LOCUS OF POINTS EQUIDISTANT FROM TWO POINTS

191. THEOREM XXXI. *The locus of all points equidistant from the extremities of a line-segment is the perpendicular bisector of the segment.*



Given the line-segment AB and its perpendicular bisector l.

To prove that every point in l is equidistant from A and B; and every point which is equidistant from A and B lies in l.

Proof: (1) Let P be any point in l. Draw PA, PB.

Then $PA = PB$. § 167, (2)

(*If from a point in a ⊥ to a line oblique segments are drawn, cutting off equal distances from the foot of the ⊥, these segments are equal and make equal ∠s with the ⊥ and with the given line.*)

(2) Let P be any point in the plane such that $PA = PB$, and let D be the middle point of AB.

Then the line PD is the perpendicular bisector of AB. § 126

(*Two points, each equidistant from the extremities of a line-segment, determine the ⊥ bisector of the segment.*)

That is, P lies on the perpendicular bisector of AB.

Hence the perpendicular bisector l of AB fulfills the two conditions:

(1) Every point in it is equidistant from A and B.

(2) Every point which is equidistant from A and B lies in it.

That is, l is the locus sought.

Q. E. D.

SIGHT WORK

1. A locomotive drive-wheel 60 inches in diameter rolls along a straight rail. What is the locus of the center of the hub of the wheel?

2. What is the locus of all points on this page equally distant from both the right-hand edge and the lower edge?

3. What is the locus of all points in a plane 2 inches from a fixed point?

4. What is the locus of the end point of a hand on the clock?

5. What is the locus of the vertices of all isosceles triangles which can be constructed on a given base?

POLYGONS

192. The Polygon and its Parts. A plane figure bounded by line-segments is called a *polygon.*

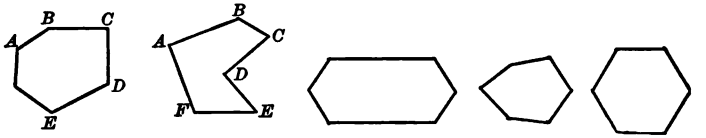
The segments are the *sides* of the polygon and the points where they meet are its *vertices.* The angles between adjacent segments are the angles of the polygon.

A segment connecting two non-adjacent vertices is a *diagonal* of the polygon.

The *perimeter* of a polygon is the sum of its sides.

193. Kinds of Polygons. A polygon is *convex* if no side when produced enters it. Otherwise it is *concave.*

Only convex polygons are considered in this text.



CONVEX CONCAVE EQUIANGULAR EQUILATERAL REGULAR

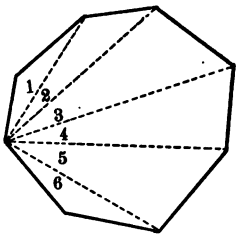
A polygon is *equiangular* if all its angles are equal, and *equilateral* if all its sides are equal.

A polygon is *regular* if it is both equiangular and equilateral.

A polygon of 5 sides is called a *pentagon*, one of 6 sides a *hexagon*, one of 8 sides an *octagon*, one of 10 sides a *decagon.*

THE SUM OF THE ANGLES OF A POLYGON

194. THEOREM XXXII. *The sum of the interior angles of a polygon of n sides is* $(2n-4)$ *right angles.*



Proof: Connect one vertex with each of the other non-adjacent vertices, thus forming a set of triangles. Evidently the sum of the angles of these triangles equals the sum of the angles of the polygon. Ax. 6, § 71

(*The whole is equal to the sum of all its parts.*)

If the polygon has n sides there will be $n-2$ such triangles. The sum of the $\measuredangle$ of $(n-2)$ $\triangle$ is $2(n-2)$ rt. $\measuredangle = (2n-4)$ rt. $\measuredangle$.

$\therefore$ The sum of the $\measuredangle$ of the polygon is $(2n-4)$ rt. $\measuredangle$.

Q. E. D.

SIGHT WORK

1. What is the value of $2n-4$, if $n=3$? if $n=4$? Hence, the sum of the angles of a quadrilateral equals how many right angles?

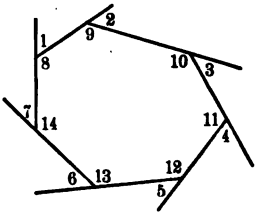
2. What is the value of $2n-4$, if $n=5$? The sum of the angles of a pentagon equals how many right angles? How many degrees in one angle of an equiangular pentagon?

3. What is the value of $2n-4$, when $n=6$? The sum of the angles of a hexagon equals how many right angles? How many degrees in one angle of an equiangular hexagon?

4. Answer questions similar to the above for a seven-sided polygon (heptagon), also for a twelve-sided polygon (dodecagon).

THE SUM OF THE EXTERIOR ANGLES OF A POLYGON

195. THEOREM XXXIII. *The sum of the exterior angles of a polygon, formed by producing the sides in succession, is four right angles.*



Given a polygon with the sides extended in succession.

To prove that the sum of the exterior ∡s thus formed is 4 rt. ∡s.

Proof: The sum of the interior and exterior angles at one vertex is 2 rt. angles. Why?

Therefore the sum of all the interior and exterior angles at n vertices is $2n$ rt. ∡s.

But the sum of the interior angles is $(2n - 4)$ rt. ∡s. § 194

(*The sum of the ∡s of a polygon having n sides is* $(2n - 4)$ *rt. ∡s.*)

∴ the sum of the exterior angles is

$$[2n - (2n - 4)] \text{ rt. } \measuredangle s = 4 \text{ rt. } \measuredangle s.$$

SIGHT WORK

1. If a polygon is equiangular how do its exterior angles compare among themselves?

2. How many degrees in an exterior angle of an equiangular pentagon?

3. How many degrees in an exterior angle of an equiangular hexagon?

EXERCISES

1. Prove Theorem XXXII, using the figure given here.

2. A surveyor measures the angles of a figure $ABCDE$ as follows: $\angle A = 73° 30'$, $\angle B = 110° 45'$, $\angle C = 98° 15'$, $\angle D = 115° 30'$, $\angle E = 141° 15'$. What is the total error? On an average did he measure the angles too large or too small? To correct his results should they be increased or decreased? Correct his results by dividing the total error by 5 and adding or subtracting the quotient as may be necessary. Corrections of this kind are always made by surveyors.

3. In a figure $ABCDEF$, the $\measuredangle A$, B, C, D, E, are found by measurement to be 58°, 145°, 93°, 134° 30′, 162° 45′. How many degrees would this leave in $\angle F$?

4. A surveyor measures the angles of a polygonal field with the following results: $\angle A = 91° 45'$, $\angle B = 150° 30'$, $\angle C = 120° 15'$, $\angle D = 134° 45'$, $\angle E = 147°$, $\angle F = 74°$. What is the total error and the average error?

EXPERIMENTAL GEOMETRY

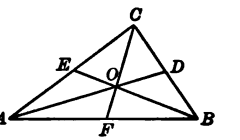
In the following constructions make the figures large and very accurate. Otherwise the conclusions obtained from them will be of little value.

1. Construct the perpendicular bisectors of the sides of a triangle. Do they meet in a point? What proposition about the perpendicular bisectors of the sides of a triangle does this seem to verify?

2. Construct the bisectors of the angles of a triangle. Do these bisectors meet in a point? What proposition about the bisectors of the angles of a triangle does this experiment seem to verify?

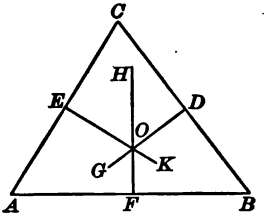
3. Construct the three altitudes of a triangle. Do they meet in a point? What proposition about the altitudes of a triangle does this seem to verify?

4. Construct a triangle and its medians and letter as in the figure. Measure AO and OD with care and compare them. Also measure BO and OE, CO and OF and compare. What proposition about these medians does this seem to verify?



THE CIRCUMCENTER OF A TRIANGLE

196. THEOREM XXXIV. *The three perpendicular bisectors of the sides of a triangle meet in a point.*



Given *FH*, *DG*, and *EK* perpendicular bisectors of the sides *AB*, *BC*, and *AC* of △ *ABC*.

To prove that FH, DG, and EK meet in a point.

Proof: *DG* and *EK* will meet in some point *O*.

Otherwise they would be parallel, which is impossible. § 154

(*If two lines are ⊥ to two intersecting lines, respectively, then the given lines cannot be* ∥.)

Then *O* is equidistant from the points *B* and *C* and also from *C* and *A*. § 191

(*The locus of all points equidistant from the extremities of a line segment is the perpendicular bisector of the segment.*)

∴ *O* is equidistant from *A* and *B*. Why?

But all points that are equidistant from *A* and *B* lie on the perpendicular bisector of *AB*. § 191

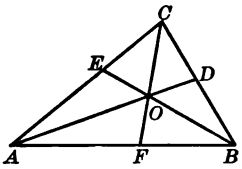
∴ *FH* passes through *O*.

∴ *DG*, *EK*, *FH* meet in the common point *O*. Q. E. D.

197. **The Circumcenter of a Triangle.** The point in which the perpendicular bisectors of the sides of a triangle meet is called the *circumcenter* of the triangle.

THE INCENTER OF A TRIANGLE

198. THEOREM XXXV. *The bisectors of the three angles of a triangle meet in a point.*



Given $\triangle ABC$ **with** AD, BE, CF **bisecting the angles** A, B, C, **respectively.**

To prove that AD, BE, *and* CF *meet in one point.*

Proof: AD and BE will meet in some point O.

(For they cannot be $\parallel$ since, in that case, $\angle DAB + \angle EBA = 2$ rt. $\measuredangle$ (§ 153, (2)), which would make $\angle A + \angle B = 4$ rt. $\measuredangle$.)

Then O is equidistant from AB and AC. § 188, (1)

(*Every point in the bisector of an* $\angle$ *is equidistant from its sides.*)

O is also equidistant from AB and BC. Why?

Therefore O is equidistant from AC and BC. Ax. 5, § 71

But all points equidistant from AC and BC lie on CF. § 188, (2)

(*If a point is equidistant from the sides of an* $\angle$, *etc.*)

$\therefore$ CF passes through O.

$\therefore$ AD, BE, CF meet in a common point O.

Q. E. D.

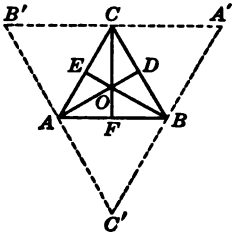
199. **The Incenter.** The point in which the bisectors of the angles of a triangle meet is called the *incenter* of the triangle.

IMPORTANT NOTE. — The remaining theorems of Book I (pages 97–103) may be used for supplementary work or omitted if desired. They are not fundamental in the same sense as the preceding theorems (I to XXXV), and their omission will not disturb the continuity of the course.

SUPPLEMENTARY THEOREMS AND PROBLEMS

THE ORTHOCENTER OF A TRIANGLE

200. THEOREM. *The three altitudes of a triangle meet in a point.*



Given $\triangle ABC$ **with altitudes** AD, BE, **and** CF.

To prove that AD, BE, *and* CF *meet in a point.*

Proof: Through each vertex of the triangle draw a line parallel to the opposite side, forming a triangle $A'B'C'$.

Then $ABA'C$ is a $\square$. § 173

$\therefore AB = CA'$. Why?

Similarly $ABCB'$ is a $\square$ and $AB = B'C$.

That is, C is the middle point of $A'B'$.

Moreover, $CF \perp A'B'$. § 151

(*A line* $\perp$ *to one of two* $\parallel$ *lines is* $\perp$ *to the other also.*)

That is, CF is the perpendicular bisector of $A'B'$.

In the same manner we show that AD is the perpendicular bisector of $B'C'$ and BE is the perpendicular bisector of $C'A'$.

$\therefore AD$, BE, CF meet in a common point. § 196

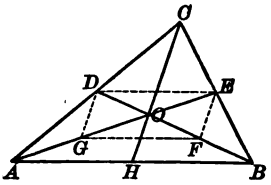
(*The three* $\perp$ *bisectors of the sides of a* $\triangle$ *meet in a point.*)

Q. E. D.

201. The Orthocenter. The point in which the three altitudes of a triangle meet is called the *orthocenter* of the triangle.

THE CENTROID OF A TRIANGLE

202. THEOREM. *The three medians of a triangle meet in a point which is two thirds of the distance from each vertex to the middle point of the opposite side.*



Given $\triangle ABC$ **with medians** BD **and** AE **meeting in** O.

To prove that the median from C *also passes through* O, *and that* $AO = \frac{2}{3} AE$, $BO = \frac{2}{3} BD$, *and* $CO = \frac{2}{3} CH$.

Proof: Let G and F be the middle points of AO and BO.

Then $GF \parallel AB$ and $GF = \frac{1}{2} AB$. §§ 184, 186

(*A segment connecting the middle points of two sides of a* $\triangle$ *is* $\parallel$ *to the third side and equal to one half of it.*)

Since D and E are the middle points of AC and BC,

$DE \parallel AB$ and $DE = \frac{1}{2} AB$. §§ 184, 186

$\therefore DE = GF$ and $DE \parallel GF$. Why?

$\therefore GFED$ is a $\square$. § 176 (2)

(*A quadrilateral is a* $\square$ *if two opposite sides are equal and* $\parallel$).

$\therefore GO = OE$ and $FO = OD$. § 177 (2)

(*In a* $\square$ *the diagonals bisect each other.*)

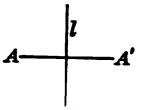
$\therefore AO = \frac{2}{3} AE$, $BO = \frac{2}{3} BD$. Why?

In the same way we may prove that CH meets BD in a point O' such that $BO' = \frac{2}{3} BD$. Hence O and O' are the same point. That is, the medians meet in a common point O. Q. E. D.

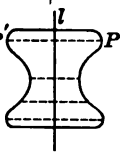
203. The Centroid. The point in which the three medians of a triangle meet is called the *centroid* of the triangle.

SYMMETRY

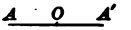
204. Axial Symmetry. Two points A and A' are **symmetric** with respect to a line l if l is the perpendicular bisector of the segment AA'.



A figure is *symmetric* with respect to an axis l, if for every point P in the figure there is a point P' also in the figure such that P and P' are symmetric points with respect to l. This is called *axial symmetry.*

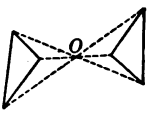


205. Central Symmetry. Two points A and A' are symmetric with respect to a point O if the segment AA' is bisected by O.



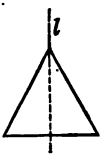
A figure is symmetric with respect to a point O if for every point P of the figure there is a point P' also in the figure such that P and P' are symmetric with respect to O.

Such a figure is said to have *central symmetry* with respect to the point O. The point is called the *center of symmetry.* A circle is a figure having central symmetry.



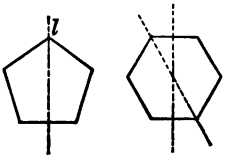
206. Symmetric Figures. Two figures are symmetric with respect to a center or with respect to an axis if, for every point A in either figure, there is a point A' in the other such that A and A' are symmetric with respect to the center or the axis.

SIGHT WORK



1. How many axes of symmetry has a square? A rectangle? Has a circle a center of symmetry? Has a circle more than one axis of symmetry?

2. How many axes of symmetry has an equilateral triangle? An isosceles triangle?

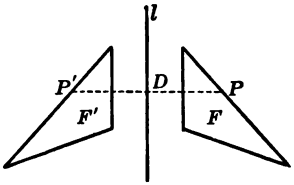


3. How many axes of symmetry has a regular pentagon (five-sided figure)?

4. How many axes of symmetry has a regular hexagon?

SYMMETRIC FIGURES ARE EQUAL

207. THEOREM. *Two figures which are symmetric with respect to a line are equal.*

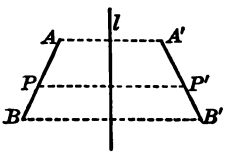


Given figures F and F' symmetric with respect to a line l.

To prove that they are equal.

Proof: By definition, for any point P of F there is a point P' of F' such that l is the perpendicular bisector of PP'. Hence, folding the figure over on the line l as an axis, every point P of F will fall upon a point P' of F'. That is, the figures will coincide and are equal.

208. COROLLARY. *If the points A and A' and also B and B' are symmetric with respect to a line l, then the segments AB and $A'B'$ are symmetric with respect to l.*



EXERCISES

1. If a line l and a point A are given, show how to find a point A' such that A and A' are symmetric with respect to l.

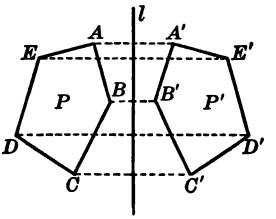
2. If a line l and a segment AB not meeting l are given, show how to construct a segment $A'B'$ such that AB and $A'B'$ shall be symmetric with respect to l.

3. Given points A and C, show how to find a point B such that C is the center of symmetry of A and B.

4. Given a segment AB and a point C not on the line AB, show how to find a segment $A'B'$ such that AB and $A'B'$ have central symmetry with C as the center.

CONSTRUCTION OF SYMMETRIC POLYGONS

209. PROBLEM. *Given a polygon P and a line l not meeting it, to construct a polygon P' such that P and P' shall be symmetric with respect to l.*



Given the polygon P and a line l not meeting it.

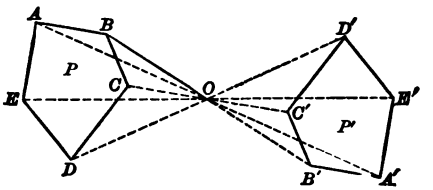
To construct a polygon P' symmetric to P with respect to l.

Construction. Let A, B, C, D, E, be the vertices of P. Construct A', B', C', D', E', symmetric respectively to A, B, C, D, E, with respect to l. Connect these points, forming a polygon P'.

Then the corresponding segments of P and P' are symmetric with respect to l. § 208

Hence, P and P' are symmetric with respect to l.

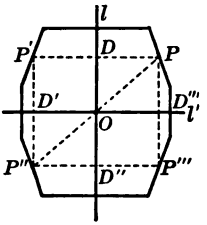
210. PROBLEM. *Given a polygon P and a point O outside of it, to construct a polygon P' symmetric to P with respect to O.*



Suggestion: Construct points symmetric to the vertices of P. Connect these points, forming the polygon P'.

TEST FOR CENTER OF SYMMETRY

211. THEOREM. *If a figure has two axes of symmetry at right angles to each other, their point of intersection is a center of symmetry of the figure.*



Given a figure with two axes of symmetry l and l' meeting at right $\measuredangle$ at O.

To prove that O is a center of symmetry of the figure.

Proof: Let P be any point of the figure. We need to prove that there is a point P'' in the figure such that the segment PP'' is bisected at O.

By hypothesis there is a point P' which is symmetric to P with respect to l. Again, there is a point P'' which is symmetric to P' with respect to l'. Let PP' cut l in D and $P'P''$ cut l' in D'. Then $PD = DP' = OD'$ and $P''D' = D'P' = OD$.

Hence, $\triangle P''D'O = \triangle ODP$. Why?

$\therefore PO = OP''$ and $\angle D'OP'' = \angle DPO$. Why?

It remains to show that PO and OP'' lie in a straight line.

Since $\angle DOP + \angle DPO = \text{rt. } \angle$,

that is, $\angle DOP + \angle D'OP'' = \text{rt. } \angle$,

it follows that $\angle D'OP + \angle D'OP'' = 2 \text{ rt. } \measuredangle$.

$\therefore P''OP$ is a straight line. § 82

Hence, PP'' is a straight line-segment bisected by the point O, and P'' and P are symmetric with respect to O.

Q.E.D.

SIGHT WORK

1. According to the theorem of § 211, has a square a center of symmetry? Has a rectangle?

2. Has an equilateral triangle a center of symmetry?

3. Has a regular pentagon a center of symmetry? See figure for Example 3, p. 99.

4. Has a regular hexagon a center of symmetry? See figure for Example 4, p. 99.

5. In each of the following figures give without proof all axes of symmetry and state whether or not the figure has a center of symmetry.

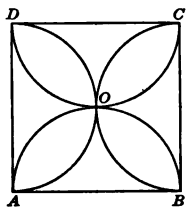


Fig. 1.

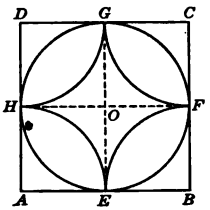


Fig. 2.

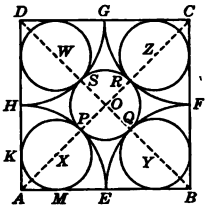


Fig. 3.

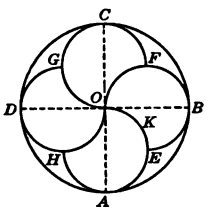


Fig. 4.

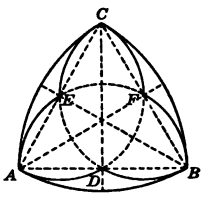


Fig. 5.

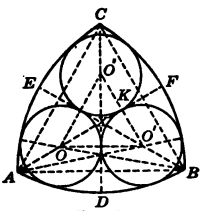


Fig. 6.

EXERCISES

1. If two non-parallel lines are symmetrical with respect to a line they meet on this line and make equal angles with it. Prove.

2. State without proof how many axes of symmetry a regular dodecagon has. Has it a center of symmetry?

3. State without proof how many axes of symmetry a regular octagon has. Has it a center of symmetry?

METHODS OF ATTACKING PROBLEMS

212. Synthetic Proof. Many theorems are proved by passing directly from the hypothesis to the conclusion by a series of logical steps. This is called a direct or *synthetic proof.*

Thus, to prove a proposition D, we may start from a known proposition A and show that a proposition B follows from it, that a proposition C follows from B, and that proposition D follows from C.

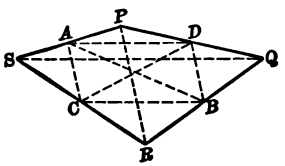
213. Analysis of Proof. Sometimes a proof may be discovered by tracing it backward from the final conclusion. This is called finding a proof by *analysis.*

Thus, to prove a proposition D, we notice that D follows from another proposition C, that C follows from proposition B, and that B follows from proposition A. If, now, A is known, D can be proved by passing from A to B, from B to C, and from C to D.

That is, the analysis is instrumental in discovering the proof. The proof itself is then made by the direct or synthetic method.

As an example, consider the proof of the following theorem:

The segments connecting the middle points of the opposite sides of any quadrilateral bisect each other.



Given segments AB and CD connecting the middle points of the sides of the quadrilateral $PQRS$.

To prove that AB and CD bisect each other.

Proof: Draw PR, SQ, AD, DB, BC, and CA.

Now AB and CD bisect each other if $ADBC$ is a $\square$, and $ADBC$ is a $\square$ if $AD \parallel CB$ and $AC \parallel DB$.

But $AD \parallel CB$ since each is $\parallel SQ$. See § 184.

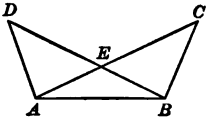
And $AC \parallel DB$ since each is $\parallel PR$.

Hence $ADBC$ is a $\square$ and AB and CD bisect each other.

Notice that the auxiliary lines PR and SQ divide the figure into triangles, and this suggests the use of § 184.

214. Further Study of Analysis. In finding a proof by analysis it may be necessary to decide by actual trial which one of several possible ways of proving any particular step will work. This is illustrated in the following examples:

Example 1. Given $AD = BC$, $\angle BAD = \angle ABC$. To prove that $CE = DE$.



Analysis: To prove two line-segments equal (as CE and DE) we may show that

(1) they are corresponding sides of equal figures, or

(2) they are sides opposite equal $\measuredangle$s in an isosceles $\triangle$, or

(3) they are opposite sides of a $\square$.

It is seen at once that (1) is the only one of these which can be used in this case. Hence we must show $\triangle BCE = \triangle ADE$.

To do this it is necessary to show that

(1) the $\triangle$s have two sides and the included $\angle$ equal, or

(2) they have two $\measuredangle$s and the included side equal, or

(3) they have three sides equal.

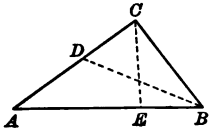
We know that $AD = BC$. Hence we try to prove $\angle D = \angle C$ and $\angle DAE = \angle EBC$. To do this we prove $\triangle BAD = \triangle ABC$.

Hence, $\angle EAB = \angle ABE$.

$\therefore \angle BAD - \angle BAE = \angle ABC - \angle ABE$, or $\angle DAE = \angle EBC$.

Hence the proof has been discovered.

Example 2. If in the figure $AB > AC$ and $EB = CD$ prove $BD > CE$.



Analysis: To prove two segments unequal we show that either

(1) they are sides of a $\triangle$ opposite unequal $\measuredangle$s, or

(2) they are the third sides of two $\triangle$s in which two sides of one are equal respectively to two sides of the other while the included $\angle$ of one is greater than the included $\angle$ of the other.

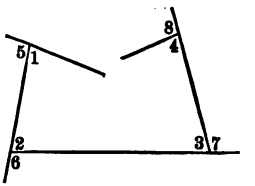
It is seen at once that only the second of these will work, namely; in the $\triangle$s BCD and CEB, $BC = BC$, $BE = CD$, and $\angle DCB > \angle EBC$ (opposite unequal sides in the $\triangle ABC$).

Hence the proof has been discovered.

215. Indirect Proof. In some cases a proof may be made most easily by assuming that the theorem is *not true* and showing that this leads to a contradiction.

As an example consider the following theorem:

A convex polygon cannot have more than three acute angles.



Proof: Assume that such a polygon may have four acute angles, as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$.

Extend the sides forming the exterior angles 5, 6, 7, 8.

Then $\angle 5$, $\angle 6$, $\angle 7$, $\angle 8$ are all obtuse, and hence $\angle 5 + \angle 6 + \angle 7 + \angle 8 > 4$ rt. $\measuredangle$ which is contrary to § 195.

Hence $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$ cannot all be acute. That is, a convex polygon cannot have more than three acute angles.

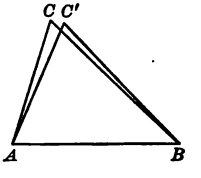
216. Method of Exclusion. The proof by the *method of exclusion* (§ 136) involves the *indirect* process in showing that all but one of the possible suppositions is false.

EXERCISES

Theorems to be proved by Indirect Proof:

1. A point within an angle and not on its bisector is unequally distant from its sides.

2. A point not in the perpendicular bisector of a segment is unequally distant from the end points of the segment.



3. Two triangles ABC and ABC' have their vertices C and C' on the same side of the common base AB. If $AC = AC'$ while the points C and C' do not coincide, prove that BC is not equal to BC'.

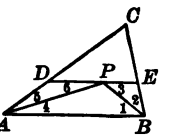
4. If the sides of triangle ABC are perpendicular respectively to the sides of a triangle $A'B'C'$, then $\angle A$ and $\angle A'$ and also $\angle B$ and $\angle B'$ cannot both be supplementary pairs of angles.

217. Problems of Construction. An effective method for solving problems of construction is to sketch the figure as if the construction were already made, and then study its properties.

For instance, consider the following problem:

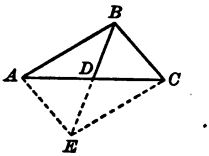
To draw a line parallel to the base of a triangle such that the segment included between the sides shall equal the sum of the segments of the sides between the parallel and the base.

Analysis: Suppose the construction made, as in the figure. If $DE = AD + BE$, there must be some point P on DE such that $AD = DP$ and $BE = EP$; that is, such that $\angle 5 = \angle 6$ and $\angle 3 = \angle 2$. Since $DE \parallel AB$, $\angle 4 = \angle 6$ and $\angle 1 = \angle 3$. Hence, $\angle 4 = \angle 5$ and $\angle 1 = \angle 2$.



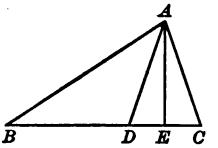
This suggests that if $\angle A$ and $\angle B$ are bisected, the bisectors will meet on the required line.

Example 1. Given two sides of a triangle and the median upon the third side, to construct the triangle.



Analysis: Suppose the triangle ABC has the required sides AB and CB, and that BD is the required median. Complete the $\square\, ABCE$. Then $AE = BC$ and $DE = BD$. Hence in the $\triangle ABE$, AB and AE are in the required sides and BE is twice the required median. The $\triangle ABE$ may therefore be constructed and then the $\triangle ABC$ is readily constructed.

Example 2. Construct a triangle ABC, having given $\angle A$, the bisector of $\angle A$, and the altitude upon the side BC.



Analysis: Suppose $\triangle ABC$ is constructed so that AE is the required altitude and AD the bisector of $\angle A$. Then $\triangle ADE$ is fixed, having given the hypotenuse and one side. Moreover $\angle DAC = \frac{1}{2} \angle A$ and $\angle DAB = \frac{1}{2} \angle A$.

218. General Suggestions for Proving Theorems.

1. *Read the theorem with care. Separate the statement into hypothesis and conclusion.*

Failure in proving a theorem is often due to careless reading of the theorem. We must know exactly what is given and exactly what is to be proved.

2. *Construct a figure, as general in form as is permitted by the hypothesis.*

Thus, if a triangle is called for, but no special form of triangle is mentioned, we should not draw an isosceles triangle or a right triangle.

If a parallelogram is specified, we should not draw a rectangle or a rhombus.

3. *Construct the figure as accurately as possible.*

This is especially important in the early part of the work. The properties of a figure are often suggested if it is accurately drawn.

4. *Recall all propositions which bear on the theorem to be proved and try them in turn.*

Thus, if it is required to prove two line-segments equal, recall every theorem whose conclusion is that two segments are equal; *i.e.* "Sides opposite equal $\measuredangle$ of a triangle are equal"; "Opposite sides of a parallelogram are equal," etc.

5. *If necessary, construct auxiliary lines.*

On this point, study theorems already proved. See, for example, §§ 130, 133.

6. *If a proof cannot be found directly, try to find it by analysis, or try an indirect proof* (see § 213).

219. General Suggestions for Solving Problems of Construction.

1. *Read the problem with care.*

2. *Sketch the figure as if the construction were already made and study its properties.* See § 217.

3. *Try to make the construction on the basis thus suggested.*

EXERCISES RELATING TO EQUAL ANGLES AND EQUAL SEGMENTS

(Pages 97–102 are not required for any exercises on pages 109–115.)

1. Make a list of all propositions in Book I in which triangles or quadrilaterals are found equal.

Such theorems are those of §§ 104, 105, 107, 113, 163, 164, etc.

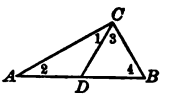
2. Make a list of all propositions in the introduction and Book I which give directly the equality of angles.

Such theorems are those of §§ 74, 77, 78, 79, 102, 109, 110, etc.

3. Make a list of all propositions in Book I which give directly the equality of line-segments.

Such theorems are those of §§ 137, 167, 177, 178, 181, etc.

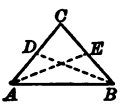
4. The middle point of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.



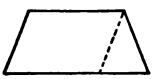
Suggestion. Draw CD so that $\angle 3 = \angle 4$. Prove $\angle 1 = \angle 2$, $CD = DB$, $CD = AD$.

5. If the median upon the base of a triangle equals half the base, the triangle is a right triangle.

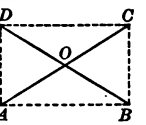
6. The medians drawn to the equal sides of an isosceles triangle are equal.



7. If the base angles of a trapezoid are equal, the non-parallel sides are equal and the trapezoid is isosceles. Also prove the converse.



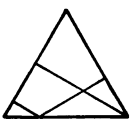
8. If the diagonals of a parallelogram are equal, the figure is a rectangle.



9. If through the extremities of the base of an isosceles triangle lines are drawn parallel to the equal sides, another isosceles triangle is formed.

Draw several isosceles triangles of different shapes to illustrate this.

10. The sum of the perpendiculars from a point in the base of an isosceles triangle to the sides is equal to the altitude drawn from an extremity of the base to the side opposite.



EXERCISES ON INEQUALITIES OF LINES OR ANGLES

1. Make a list of all propositions in Book I in which lines are found unequal.

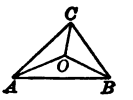
Such propositions are those of §§ 71 (Axioms 7, 8, 9, 10, 11, 12, 13, and 16), 135, 139, 141, 169, etc.

2. Make a list of all propositions in the Introduction and in Book I in which angles are found unequal.

Such propositions are those of §§ 71 (Axioms 7, 8, 9, 10, 11, 12, and 13), 80, 130, 132, 133, 140, etc.

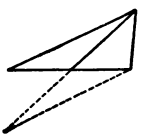
3. Show that any side of a triangle is greater than the difference between the other two sides.

4. Show that the sum of the distances from any point within a triangle to the vertices is greater than one half the sum of the sides of the triangle.



5. Show that the segment joining the vertex of an isosceles triangle to any point in the base is less than either of the equal sides.

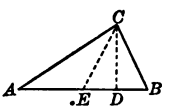
6. In any triangle the sum of two sides is greater than twice the median on the third side.



7. If the diagonals of a parallelogram are unequal, then it is not a rectangle.

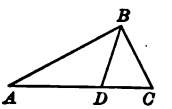
8. The sum of the sides of any quadrilateral is greater than the sum of the diagonals and less than twice their sum.

9. In a triangle ABC, $AC > BC$, $CD \perp AB$, and CE is the median on AB. Prove that: (1) $\angle ACD > \angle BCD$; (2) $\angle AEC > \angle BEC$.



Suggestions. (1) $AD > BD$, § 168 (2); (2) E lies between A and D; (3) $\angle BEC <$ rt $\angle$; (4) $\angle AEC >$ rt $\angle$.

10. If, in the figure, BD bisects $\angle B$ of the $\triangle ABC$, then show that $BC > DC$ and $BA > AD$.



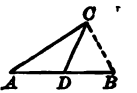
EXERCISES INVOLVING PROBLEMS OF CONSTRUCTION

1. Construct an isosceles triangle, having given its base and vertex angle.

Suggestion. Find the base angles.

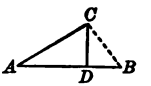
2. Construct the altitudes of an obtuse triangle. Do they meet inside or outside the triangle?

3. Construct a triangle, having given two sides and the median to one of them.



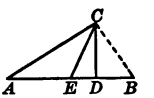
Suggestion. The sides AB, AC and median CD are given. Hence, $\triangle ADC$ is determined.

4. Construct a triangle having given two sides and the altitude on one of them.



Analysis. Suppose the construction made. AB and AC are the given sides and CD the given altitude. The triangle can be constructed if $\triangle CDA$ can be constructed. See Example 1, page 71.

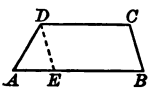
5. Construct a triangle, having given the base and the median and altitude on it.



Analysis. Suppose the construction made. $\triangle CDE$ is a rt. $\triangle$ with CE and CD given.

6. Construct a triangle, having given one side, an adjacent angle, and the altitude on the given side.

7. Construct a trapezoid having given its four sides.



Suggestion. $\triangle ADE$ is fixed.

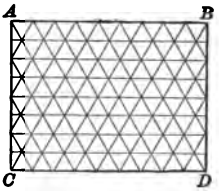
8. Through a point not on a given line to construct a line making an angle of 60° with the given line.

9. Given three points not in a straight line, to construct an equilateral triangle such that one of the given points will lie on each side of the triangle or on a side produced.

NOTE. — A large variety of problems of construction may be obtained by giving the various possible combinations of sides, angles, altitudes, medians, and angle bisectors of a triangle.

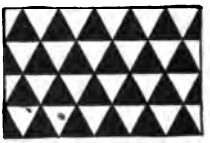
EXERCISES INVOLVING THE ANGLES OF A POLYGON

1. How many degrees in each angle of an equilateral triangle? How many equilateral triangles may be made to meet in a point and to cover the plane about it?

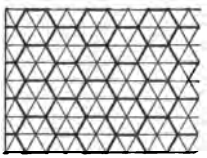


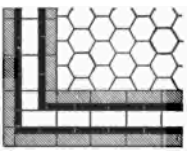
2. How many equilateral triangles may be used to form a regular hexagon? How many degrees in an angle of a regular hexagon?

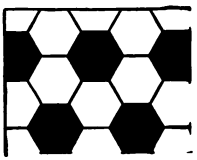
3. Is it possible to completely cover the plane with regular hexagons? Explain.



Tile Flooring.

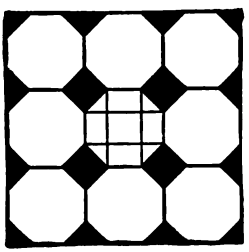






4. How many degrees in an angle of a regular pentagon? Can 3 regular pentagons be made to meet at a point without overlapping? Will they completely cover the plane about the point? Can four regular pentagons be made to meet at a point without overlapping? Is it possible to cover a plane with non-overlapping pentagons?

5. How many degrees in an angle of a regular octagon (eight-sided figure)?



6. Is it possible to cover a plane with non-overlapping regular octagons?

7. If two regular octagons and a square are made to meet in a point, will the plane be covered without overlapping? Study the adjoining figure.

EXERCISES INVOLVING PERPENDICULARS AND PARALLELS

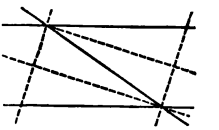
1. Make a list of all propositions in Book I involving perpendiculars.

2. Make a list of propositions involving parallels.

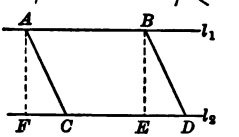
3. The diagonals of a rhombus are perpendicular to each other.

4. If the angles adjacent to a base of a trapezoid are equal, the remaining two angles are equal.

5. The bisectors of two consecutive angles of a parallelogram meet at right angles.



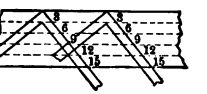
6. The bisectors of the four interior angles formed by a transversal cutting two parallel lines form a rectangle.



7. If in the figure l_1 and l_2 are parallel and $BD = AC$, then $ABDC$ is a parallelogram or an isosceles trapezoid.

Suggestion: Draw AF and $BE \perp l_2$. Prove $\triangle AFC = \triangle BED$ and $\therefore AC \parallel BD$ in this figure. Draw the figure for the trapezoid case.

8. A carpenter divides a board into strips of equal width as follows: Suppose five strips are desired. He places his square in two positions, as indicated in the figure, at such angles that the distance in inches diagonally across the board shall be some multiple of five (in the figure this distance is 15 inches). He marks the points and connects them as shown in the figure. Is this method correct? Prove.



9. The T-square shown in the figure can be set to form any angle. Show how two parallel lines may be drawn in any direction on a sheet fastened to the drawing board. (See Example 6, p. 64.)

EXERCISES INVOLVING LOCI

Answer the following without giving proofs:

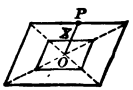
1. Find the locus of the vertex of a triangle having a fixed base and a given altitude.

2. A circle of radius one half inch is made to roll on the outside of a circle of one-inch radius. Find the locus of its center.

3. Find the locus of the middle points of the segments joining a vertex of a triangle to all points on the opposite side.

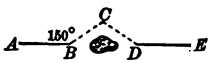
4. Given a line l and a point P not in the line. Find the locus of the middle points of all segments drawn from P to l.

5. Find the locus of the middle points of all segments joining the center (point where the diagonals meet) of a parallelogram to points on the sides.

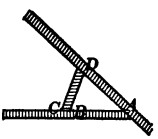


MISCELLANEOUS EXERCISES

1. In the figure show how to lay off BC and CD around an obstacle so as to make AB and DE lie on the same straight line, if the angle at B is 150°.

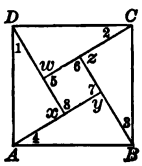


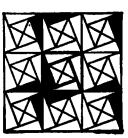
2. To cut braces for a roof, as shown in the figure, a carpenter needs to know the angle DBC when the angle DAB is given, it being given that $AB = AD$. Show how to find this angle. (See Ex. 2, page 70.)



3. If from two points in the base of an isosceles triangle parallels to the other sides are drawn, two parallelograms are formed whose perimeters are equal.

4. $ABCD$ is a square. Lines are drawn as shown in the figure so that $\angle 1 = \angle 2 = \angle 3 = \angle 4$. Prove that $xyzw$ is a square.

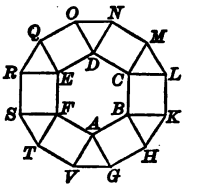




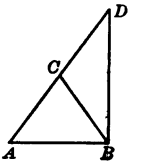
5. In the figure, *ABCDEF* is a regular hexagon. *ABHG*, *BCLK*, etc., are squares.

(*a*) What kind of triangles are *BHK*, *CLM*, etc.? Prove.

(*b*) Is the dodecagon (twelve-sided polygon) *HKLMN*... regular? Prove.

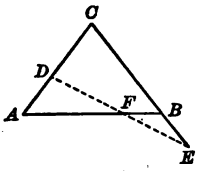


6. If *ABC* is an isosceles triangle and if *AC*, one of the equal sides, is extended its own length to *D* then *DB* is perpendicular to *AB*.

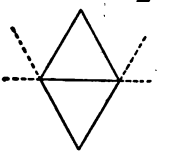


7. The bisectors of the base angles of an isosceles triangle form an angle equal to an exterior angle at the base of the triangle.

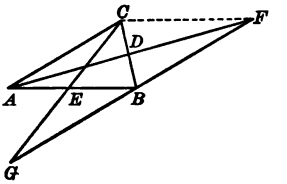
8. If *ABC* is an isosceles triangle and if *AD* is laid off on the side *AC* equal to *BE* on the side *CB* extended, and if the segment *DE* meets the base *AB* in the point *F*, then *DF* = *FE*.



9. The bisectors of the exterior angles at the base of an isosceles triangle form another isosceles triangle. Are the triangles equal?



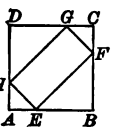
10. In a triangle *ABC*, let *E* and *D* be the middle points of *AB* and *BC* respectively. Draw *AD* and extend it to *F* making *DF* = *AD*. Also draw *CE* and extend it to *G*, making *EG* = *CE*. Show that *F*, *B*, and *G* lie in a straight line.



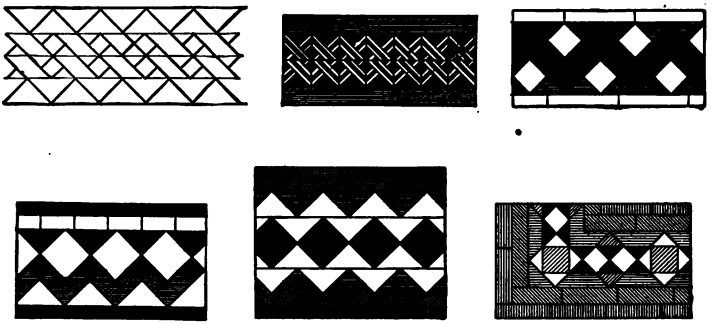
Suggestion. Prove $BF \parallel AC$ and $BG \parallel AC$.

11. If the pairs of opposite sides of a hexagon are equal and parallel, then the lines joining pairs of opposite vertices will all pass through a common point.

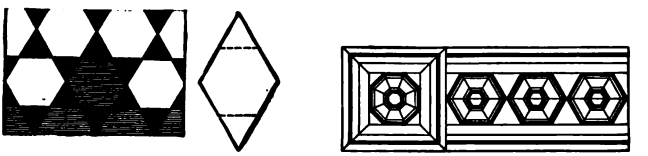
12. If equal distances are laid off on the sides of a square from two opposite vertices, the figure formed by joining these points is a rectangle.

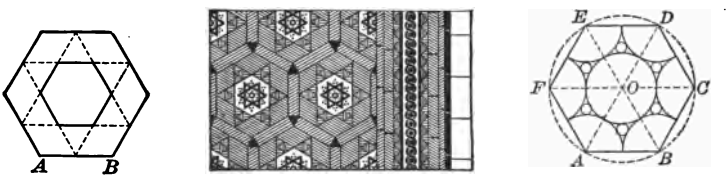


DESIGNS BASED ON PARALLEL LINES

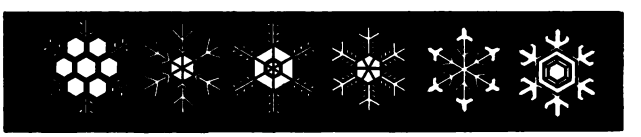


DESIGNS BASED ON EQUILATERAL TRIANGLES AND HEXAGONS





REGULAR HEXAGONS SHOWN IN SNOW CRYSTALS



Blaise Pascal was born at Clermont, France, in 1623, and died at Paris in 1662. He was a celebrated philosopher and mathematician.

At the age of twelve years, Pascal was found drawing charcoal figures on the pavement and proving theorems in geometry. At the age of sixteen he wrote a treatise on algebraic geometry which displayed great power. At nineteen he invented a machine for performing arithmetical operations, — the forerunner of our modern calculating machines.

In 1653 Pascal devised a scheme for writing down the coefficients of the binominal expansion, which he called the "Arithmetical Triangle," and which has ever since been known as the "Pascal Triangle."

Pascal wrote extensively on both philosophy and mathematics.

BOOK II

STRAIGHT LINES AND CIRCLES

220. Definition of Circle. A circle was defined in § 23, as a closed *plane curve* all points of which are equally distant from a point in the plane called the *center.*

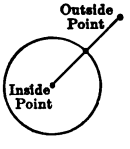
221. Circle Defined as a Locus. A circle may also be defined as the locus of all points in a plane at a fixed distance from a given point in the plane.

222. Circumference of Circle. The length of a circle, that is, the distance around it, is called the *circumference* of the circle.[1]

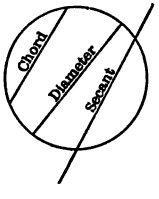
223. Notation for Circle. A circle may be denoted by two letters, one at the center and one marking a point on the circle.

Thus, ⊙ *OA* means *the circle with center O and radius OA.*

When no ambiguity arises, the letter at the center alone may be used to denote the circle. Thus, ⊙ *C* means *the circle whose center is C.*



224. Inside and Outside of Circle. A circle divides the plane into two parts such that any point which does not lie on the circle lies *within* it or *outside* it.



225. Chord. Diameter. A line-segment joining two points on a circle is called a *chord.* A chord which passes through the center is a *diameter.*

226. Secant. If a chord is extended in one or both directions, it *cuts* the circle and is called a *secant.*

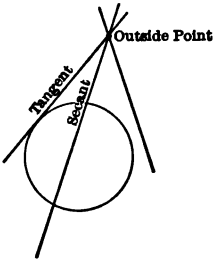
---

[1] NOTE. This is the sense in which the words *circle* and *circumference* are used in most higher mathematical works. The word *circle*, however, is sometimes used to mean *the portion of the plane bounded by the curve.*

DEFINITIONS RELATING TO CIRCLES

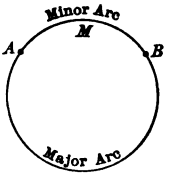
227. Tangent. A *tangent* to a circle is a straight line which meets it in one and only one point, no matter how far the line is extended.

An indefinite straight line through a point outside a circle is a *secant*, a *tangent*, or *it does not meet the circle.*



228. Arc. The portion of a circle included between any two of its points is called an *arc.* An arc AB is denoted by the symbol $\overset{\frown}{AB}$.

229. Semicircle. A circle is divided into two arcs by any two of its points. If these arcs are equal, each is a *semicircle.* Otherwise one is called the **major arc** and the other the **minor arc.**



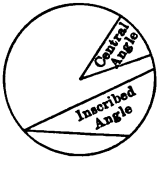
Unless otherwise indicated $\overset{\frown}{AB}$ means the *minor* arc. In case of ambiguity a third letter may be used, as arc AMB or $\overset{\frown}{AMB}$.

230. Subtended Arc. An arc is said to be *subtended* by the chord which joins its end-points.

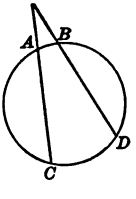
Unless otherwise indicated the arc subtended by a chord means the *minor* arc.

231. Central Angle. An angle formed by two radii is called a *central angle.*

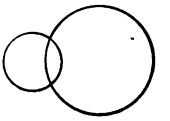
232. Inscribed Angle. An angle formed by two chords drawn from the same point on the circle is called an *inscribed angle.*



233. Intercepted Arcs. If the sides of an angle meet a circle, the arc or arcs which lie within the angle are called *intercepted arcs.*

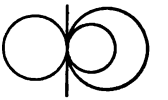


If the vertex of the angle is within or on the circle there is only one intercepted arc; if it is outside the circle there are two intercepted arcs, as $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ in the figure.



234. Intersecting Circles. If a circle is partly inside and partly outside another circle, then the circles cut each other and are said to intersect.

235. Tangent Circles. Two circles are said to be *tangent to each other* if both are tangent to a straight line at a common point. They are *tangent externally* if each circle is *outside* the other; and internally if one circle is *inside* the other.



THEOREMS PROVED INFORMALLY

The following propositions require no formal proof.

236. *If the radii or diameters of two circles are equal, the circles may be made to coincide and are equal.*

237. *A diameter of a circle is double the radius.*

238. *A point lies within, outside, or on a circle, according as its distance from the center is less than, greater than, or equal to the radius.*

239. *If an unlimited straight line contains a point within a circle, then it cuts the circle in two points.*

240. *If two circles intersect once, they intersect again. See figure,* § 234.

241. *If two minor arcs (or two major arcs) of the same or of equal circles are so placed that their end-points coincide and also their centers, then the arcs coincide throughout.*

242. *If two circles are made to coincide, then their centers coincide.*

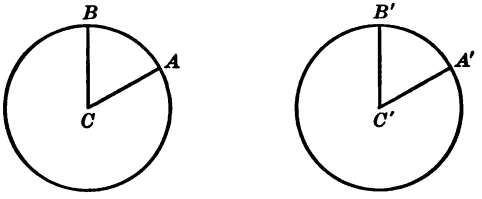
243. *If in two circles an arc of one is equal to an arc of the other, then not only the arcs but also the circles can be made to coincide throughout.*

EQUAL ARCS AND EQUAL CENTRAL ANGLES

244. THEOREM I. *In the same circle or in equal circles,*

(1) *equal central angles intercept equal arcs; and*

(2) *conversely, equal arcs are intercepted by equal central angles.*



(1) **Given** $\odot C = \odot C'$ **and** $\angle C = \angle C'$.

To prove that $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Proof: Place $\odot C$ on $\odot C'$ making $\angle C$ coincide with $\angle C'$.

Then A will fall on A' and B on B'. § 85

(*Radii of the same circle or of equal circles are equal.*)

Hence $\overset{\frown}{AB} = \overset{\frown}{A'B'}$. § 241

(*If two minor arcs (or two major arcs) of the same ⊙ or of equal Ⓢ are so placed that their end-points coincide and also their centers, then the arcs coincide throughout.*)

(2) **Given** $\odot C = \odot C'$ **and** $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

To prove that $\angle C = \angle C'$.

Proof: Place $\odot C$ on $\odot C'$ making $\overset{\frown}{AB}$ coincide with $\overset{\frown}{A'B'}$. § 243

(*If in two Ⓢ an arc of one is equal to an arc of the other, then not only the arcs but also the Ⓢ can be made to coincide throughout.*)

Then C will fall on C'. § 242

(*If two Ⓢ are made to coincide, then their centers coincide.*)

Since A falls on A', B on B', and C on C'.

$\therefore \angle C = \angle C'$. § 98

Q. E. D.

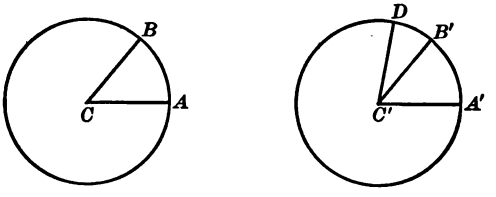
245. COROLLARY. *A diameter bisects a circle.*

Suggestion. In this case the equal central ∡s are straight ∡s.

UNEQUAL ARCS AND UNEQUAL CENTRAL ANGLES

246. THEOREM II. *In the same circle or in equal circles,* (1) *if two central angles are unequal, the greater angle intercepts the greater arc; and*

(2) *conversely, if two arcs are unequal, the greater arc is intercepted by the greater central angle.*



(1) **Given** $\odot C = \odot C'$, **and** $\angle A'C'D > \angle ACB$.

To prove that $\overset{\frown}{A'D} > \overset{\frown}{AB}$.

Proof: Apply $\odot C$ to $\odot C'$ making CA coincide with $C'A'$. Then CB will fall within the $\angle A'C'D$ and B will fall at some point B' on the intercepted arc $A'D$.

($\angle A'C'D$ *is given greater than* $\angle ACB$.)

$\therefore \overset{\frown}{A'D} > \overset{\frown}{A'B'}$, and $\overset{\frown}{A'D} > \overset{\frown}{AB}$. Why?

(2) **Given** $\odot C = \odot C'$ **and** $\overset{\frown}{A'D} > \overset{\frown}{AB}$.

To prove that $\angle A'C'D > \angle ACB$.

Proof: Apply $\odot C$ to $\odot C'$ making the radius CA coincide with its equal $C'A'$ and making $\overset{\frown}{AB}$ fall along $\overset{\frown}{A'D}$.

Then B will fall at some point B' on the arc $A'D$,

($\overset{\frown}{A'D}$ *is given greater than* $\overset{\frown}{AB}$.)

and CB will fall within $\angle A'C'D$.

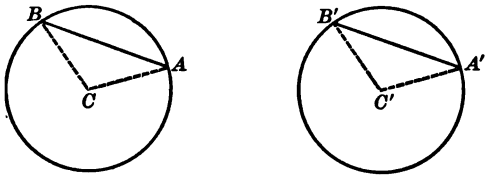
$\therefore \angle A'C'D > \angle A'C'B'$, and $\angle A'C'D > ACB$. Why?

Q. E. D.

EQUAL CHORDS AND EQUAL ARCS

247. THEOREM III. *In the same circle or in equal circles,* (1) *equal chords subtend equal arcs; and*

(2) *conversely, equal arcs are subtended by equal chords.*



(1) **Given** $\odot C = \odot C'$ **and chord** $AB =$ **chord** $A'B'$.

To prove that $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Analysis: Draw the radii CA, CB, $C'A'$, $C'B'$.

In order to prove that $\overset{\frown}{AB} = \overset{\frown}{A'B'}$, we need to show that $\angle C = \angle C'$, and to do this we show $\triangle ABC = \triangle A'B'C'$.

Proof: Since $AB = A'B'$, $CB = C'B'$, $CA = C'A'$; then $\triangle ABC = \triangle A'B'C'$, and $\angle C = \angle C'$. Why?

$\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$. § 244 (1)

(*In the same* $\odot$ *or in equal* ⑤ *equal central* ∡*s etc.*)

(2) **Given** $\odot C = \odot C'$ **and** $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

To prove that chord AB = chord $A'B'$.

Analysis: We can prove chord AB = chord $A'B'$ if we prove $\triangle ABC = \triangle A'B'C'$.

Proof: $\angle C = \angle C'$. § 244 (2)

(*In the same* $\odot$ *or in equal* ⑤ *equal arcs are intercepted etc.*)

Also $CB = C'B'$ and $CA = C'A'$. Why?

$\therefore \triangle ABC = \triangle A'B'C'$, and chord AB = chord $A'B'$. Why?

Q. E. D.

SIGHT WORK

In giving the definitions required below, make sure that your statement says exactly what you mean. Accuracy of statement is one of the most essential elements in the good use of English.

1. Define a circle as a locus.

2. Define circumference, center.

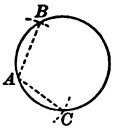
3. Define chord, radius, diameter, secant, tangent.

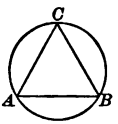
4. Define arc, semicircle, arc subtended by a chord.

5. Define central angle, inscribed angle.

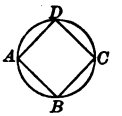
6. What is meant by the statement, "An arc is intercepted by an angle?"

7. Two arcs AB and AC are laid off on the same circle by striking arcs with equal radii using A as a center. Compare arcs AB and AC.



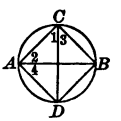


8. In the figure to the left measure and compare the chords AB, BC, and CA. Compare the three arcs subtended by them.



9. In the figure to the right measure and compare the chords AB, BC, CD, and DA. Compare the four arcs subtended by them.

10. Show how to bisect an arc, using § 244 (1).

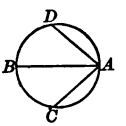


11. Show that if two diameters, AB and CD, are perpendicular to each other, they divide the circle into four equal arcs.

12. Using the figure of Example 11, show that chords AC, CB, BD, and DA are all equal and that $\angle 1 = \angle 2 = \angle 3 = \angle 4$.

13. Show how to divide a circle into six equal arcs.

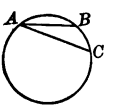
14. Show how to divide a circle into eight equal arcs.



15. In the figure AB is a diameter. Chord AD = chord AC. Compare $\overset{\frown}{DB}$ and $\overset{\frown}{BC}$.

EXPERIMENTAL GEOMETRY

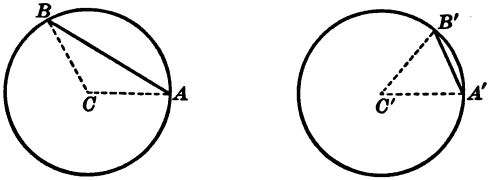
1. On a circle lay off arcs AB and AC each less than a semicircle, so that $\overset{\frown}{AC} > \overset{\frown}{AB}$. Draw chords AB and AC. Measure these chords to compare their lengths. State a proposition which seems to hold concerning the chords which subtend two unequal arcs, each less than a semicircle.



UNEQUAL CHORDS AND UNEQUAL ARCS

248. THEOREM IV. *In the same circle or in equal circles;* (1) *if two chords are unequal the greater chord subtends the greater minor arc; and*

(2) *conversely, if two minor arcs are unequal the greater arc is subtended by the greater chord.*



Given $\odot C = \odot C'$ **and chord** $AB >$ **chord** $A'B'$.

To prove that $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

(1) **Analysis:** $\overset{\frown}{AB} > \overset{\frown}{A'B'}$ if $\angle C > \angle C'$.

Proof: Draw radii CA, CB, $C'A'$, $C'B'$.

In $\triangle ABC$ and $A'B'C'$, $AB > A'B'$, $CA = C'A'$ and $CB = C'B'$.

$\therefore \angle C > \angle C'$. § 140

(*If in two* $\triangle$ *two sides of one are equal to two sides of the other, etc.*)

$\therefore \overset{\frown}{AB} > \overset{\frown}{A'B'}$. § 246, (1)

(*In the same* $\odot$ *or in equal* ⑤ *if two central* $\angle$*s are unequal, etc.*)

(2) **Given** $\odot C = \odot C'$ **and** $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

To prove that chord $AB >$ chord $A'B'$.

Proof: Since $\overset{\frown}{AB} > \overset{\frown}{A'B'}$ it follows that $\angle C > \angle C'$. § 246 (2)

(*In the same* $\odot$ *or in equal* ⑤ *if two arcs are unequal, etc.*)

In $\triangle ABC$ and $A'B'C'$, $CA = C'A'$, $CB = C'B'$ and $\angle C > \angle C'$.

$\therefore$ chord $AB >$ chord $A'B'$. § 139

(*If in two* $\triangle$ *two sides of one are equal to two sides of the other, etc.*)

Q. E. D.

249. COROLLARY. *In the same circle or in equal circles:*

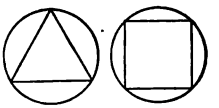
(1) *If two chords are unequal, the greater chord subtends the lesser major arc; and*

(2) *Conversely, if two major arcs are unequal, the greater arc is subtended by the lesser chord.*

250. **Inscribed Polygon. Circumscribed Circle.** A polygon is *inscribed* in a circle if its vertices lie on the circle. The circle is said to be *circumscribed* about the polygon.

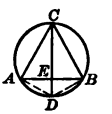
EXERCISES

1. If an equilateral triangle and a square are inscribed in the same circle, or in equal circles, show that the sides of the triangle are greater than those of the square.



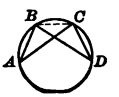
2. Prove that two diameters of a circle divide it in two pairs of equal arcs.

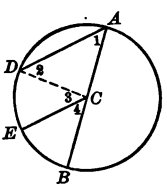
3. If ABC is an isosceles triangle inscribed in a circle, and if CD is perpendicular to AB, prove that $\overset{\frown}{AD} = \overset{\frown}{BD}$.



Suggestion. Prove that chord AD = chord BD.

4. If in the figure $\overset{\frown}{AB} = \overset{\frown}{CD}$, prove that chord AC = chord BD, and that $\triangle ABC = \triangle DCB$.

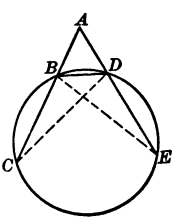




5. In the figure AB is a diameter, AD is any chord drawn from A, and CE is a radius parallel to AD. Prove $\overset{\frown}{DE} = \overset{\frown}{EB}$.

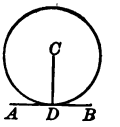
Suggestion. Prove $\angle 1 = \angle 2 = \angle 3 = \angle 4$.

6. If in the figure $\overset{\frown}{BC} = \overset{\frown}{DE}$, prove that $\triangle ABD$ is an isosceles triangle.



Suggestion. Prove (1) $\triangle CBD = \triangle EDB$, (2) $\angle CBD = \angle EDB$, (3) $\angle ABD = \angle ADB$. (See Example 4.)

EXPERIMENTAL GEOMETRY



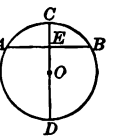
1. At the extremity D of a radius CD, construct a line $AB \perp CD$. Does this line meet the circle except at D? What is this line called?

Complete the following proposition: "A line perpendicular to a radius at its extremity is . . ."

2. Using the same figure as in the preceding example, draw any line through D, other than AB. Does such a line meet the circle in any point besides D? How many tangents to a circle may be drawn through any one point on it?

3. Draw a line which meets a circle in only one point (a tangent). At the point of contact construct a perpendicular to this line. Does this perpendicular pass through the center of the circle? Formulate this fact as a proposition.

4. How are the propositions in Examples 1 and 3 related?



5. Construct a diameter perpendicular to a chord of a circle. Measure and compare the segments into which the chord is divided.

6. In the figure constructed in Example 5 what is the relation between $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$?

Suggestion. Measure the chords and use § 247 (2).

7. Complete the proposition: "A diameter perpendicular to a chord bisects . . ."

8. Connect the center of a circle with the middle point of a chord. Measure and compare the angles thus formed. Complete the proposition: "A diameter which bisects a chord is . . ."

9. What is the relation between the propositions in Examples 5 and 8?

10. Construct the perpendicular bisector of a chord. Does it pass through the center of the circle? Formulate this fact as a proposition.

11. What is the relation between the propositions in Examples 8 and 10?

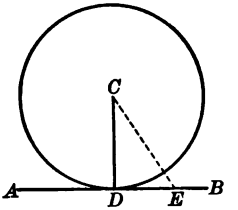
12. Construct two equal chords of a circle. Measure and compare their distances from the center of the circle. Formulate this result as a proposition. State the converse of this proposition.

13. Construct two unequal chords of a circle. Compare their distances from the center of the circle. Formulate this result as a proposition. State the converse of this proposition.

Do the measurements and comparisons made above prove these propositions, or only make them seem plausible?

TANGENT AND RADIUS

251. THEOREM V. *A line perpendicular to a radius at its extremity is tangent to the circle.*



Given $\odot C$ with $AB \perp$ to the radius CD at D.

To prove that AB *is tangent to the* $\odot$.

Proof: Let E be any point of AB other than D. Draw CE.

Then $CE > CD$. § 169

(*The* $\perp$ *is the shortest distance from a point to a line.*)

Hence E is outside the circle. § 238

(*A point lies within, outside, or on a* $\odot$ *according as its distance, etc.*)

That is, every point of AB except D is outside the circle.

$\therefore AB$ is tangent to $\odot C$. § 227

(*A tangent to a* $\odot$ *is a straight line which meets it, etc.*)

Q. E. D.

252. COROLLARY 1. *If a line is tangent to a circle, the radius drawn to the point of contact is perpendicular to the tangent.*

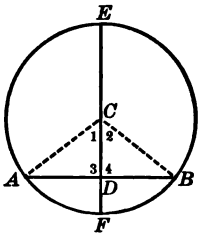
Suggestion. If CD were oblique to AB, then some other line through C as CE would be $\perp AB$. This would make $CE < CD$ and therefore E would be inside the circle and AB would not be a tangent.

253. COROLLARY 2. *A line perpendicular to a tangent at the point of contact passes through the center of the circle.*

254. COROLLARY 3. *A perpendicular to a tangent from the center of the circle passes through the point of contact.*

DIAMETER PERPENDICULAR TO A CHORD

255. THEOREM VI. *A diameter perpendicular to a chord bisects the chord and also both the subtended arcs.*



Given the diameter $EF \perp AB$ **at** D.

(1) *To prove that* $AD = DB$, *and* $\overset{\frown}{AF} = \overset{\frown}{FB}$ *and* $\overset{\frown}{AE} = \overset{\frown}{EB}$.

Analysis: Draw radii CA and CB.

If it can be shown that $\triangle ADC = \triangle BDC$, then

(*a*) $AD = DB$, (*b*) $\angle 1 = \angle 2$, (*c*) $\angle ACE = \angle BCE$,
(*d*) $\overset{\frown}{AF} = \overset{\frown}{FB}$, (*e*) $\overset{\frown}{AE} = \overset{\frown}{EB}$.

Proof: In $\triangle ADC$ and BDC, $CD = CD$, $CA = CB$. Why?

$\therefore \triangle ADC = \triangle BDC$. § 163

(*Two rt. $\triangle$ are equal if the hypotenuse and a side of one, etc.*)

Now give reasons for the following statements:

(*a*) $AD = DB$, (*b*) $\angle 1 = \angle 2$, (*c*) $\angle ACE = \angle BCE$,
(*d*) $\overset{\frown}{AF} = \overset{\frown}{FB}$, (*e*) $\overset{\frown}{AE} = \overset{\frown}{EB}$.

256. COROLLARY 1. *A diameter which bisects a chord is perpendicular to it.*

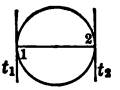
Suggestion. To prove $\angle 3 = \angle 4$, show that $\triangle ADC = \triangle BDC$.

257. COROLLARY 2. *The perpendicular bisector of a chord passes through the center of the circle.*

Suggestion. Use § 191.

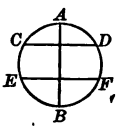
258. Concentric Circles. Two circles which have the same center are said to be *concentric.*

SIGHT WORK



1. What type of proof is suggested in § 252 ?

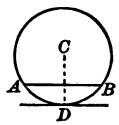
2. Show that tangents drawn to a circle at the opposite ends of a diameter are parallel.



3. Show that if a diameter bisects one of two parallel chords it bisects the other also.

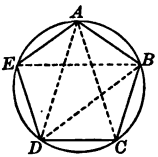
4. Show that a diameter bisects all chords parallel to the tangents at its extremities.

5. Show that the middle points of parallel chords all lie on a diameter.



6. Show that a tangent to a circle at the mid-point of any arc is parallel to the chord of the arc.

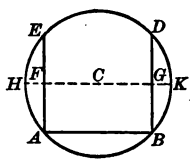
7. If, in the figure, $AB = BC = CD = DE = EA$, show that the diagonals are all equal.



8. If a circle is divided into any number of equal parts and the division points are joined in succession, the polygon thus formed will be equilateral.

9. Show that the perpendicular bisectors of all sides of a polygon inscribed in a circle meet in a point.

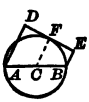
EXERCISES



1. Two chords perpendicular to the same chord at its extremities are equal.

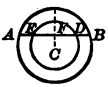
Suggestion. Draw the diameter $HK \perp AE$ and show that $\frac{1}{2}\ AE = \frac{1}{2}\ BD$ and $\therefore\ AE = BD$.

2. If, from the extremities of a diameter, perpendiculars are drawn to a tangent, the sum of these perpendiculars is equal to the diameter of the circle.



Suggestion. Draw a radius $\perp$ to the tangent and apply § 187.

3. If two concentric circles are cut by a line, the two segments intercepted between the circles are equal.

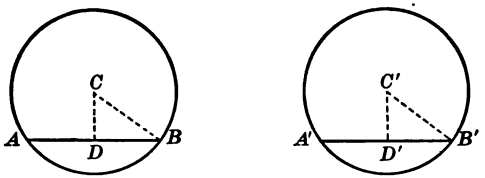


Suggestion. Draw a radius perpendicular to the line AB. Then prove (1) $AF = FB$ and (2) $EF = FD$.

CHORDS EQUALLY DISTANT FROM THE CENTER

259. THEOREM VII. *In the same circle or in equal circles;* (1) *if two chords are equal they are equally distant from the center; and*

(2) *conversely, if two chords are equally distant from the center they are equal.*



(1) **Given** $\odot C = \odot C'$, **chord** $AB =$ **chord** $A'B'$, $CD \perp AB$, $C'D' \perp A'B'$.

To prove that $CD = C'D'$.

Proof: Draw CB and $C'B'$.

Then, $DB = \frac{1}{2} AB$ and $D'B' = \frac{1}{2} A'B'$. § 255

(*A diameter $\perp$ to a chord bisects the chord, etc.*)

But $AB = A'B'$. $\therefore DB = D'B'$. Ax. 4, § 71

(*If equals are divided by equals, etc.*)

Now give reasons for the following statements:

(*a*) $BC = B'C'$, (*b*) $\triangle BCD = \triangle B'C'D'$, (*c*) $CD = C'D'$.

(2) **Given** $CD = C'D'$, $CD \perp AB$, **and** $C'D' \perp A'B'$.

To prove that *chord* $AB =$ *chord* $A'B'$.

Proof: Give reasons for the following statements:

(*a*) $CD = C'D'$, (*b*) $BC = B'C'$, (*c*) $\triangle BCD = \triangle B'C'D'$, (*d*) $BD = B'D'$.

But $AB = 2\,BD$ and $A'B' = 2\,B'D'$.

$\therefore AB = A'B'$. Ax. 3, § 71

Q. E. D.

CHORDS UNEQUALLY DISTANT FROM THE CENTER

260. THEOREM VIII. *In the same circle or in equal circles; if two chords are unequal, the greater chord is nearer the center; and*

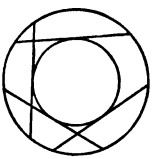
(2) *conversely, if two chords are unequally distant from the center, the one nearer the center is the greater.*

NOTE. This theorem may be proved at some length by use of § 139, but it follows as a corollary from § 352 in Book III, and its proof is deferred till that point is reached.

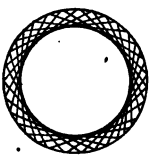
261. COROLLARY. *A diameter of a circle is greater than any other chord.*

Suggestion. The distance of a diameter from the center is less than that of any other chord.

EXERCISES

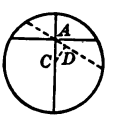


1. In two concentric circles, all chords of the larger circle which are tangent to the smaller are equal in length.



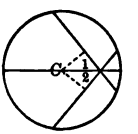
2. If in a circle many equal chords are drawn they give the effect of a circle within the given circle. Explain this.

3. The shortest chord which can be drawn through a point is perpendicular to the diameter through that point.



Suggestion. Prove that any other chord is nearer the center.

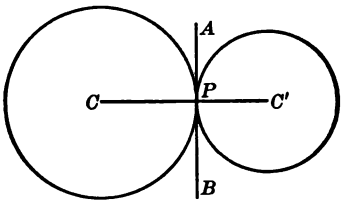
4. If two chords make equal angles with the diameter through their point of intersection, the chords are equal.



Suggestion. Prove that the chords are equally distant from the center.

TANGENT CIRCLES MEET ON THE LINE OF CENTERS

262. THEOREM IX. *If two circles meet on their line of centers, they are tangent to each other; and conversely, if two circles are tangent to each other, their line of centers passes through the point of contact.*



(1) **Given** ⓢ C **and** C' **meeting at** P **on the line** CC'.

To prove that ⓢ C *and* C' *are tangent to each other at* P.

Proof. Draw $AB \perp CC'$ at point P.

Then AB is tangent to ⊙ C and also to ⊙ C'. § 251

$\therefore$ ⊙ C is tangent to ⊙ C' at P. § 235

(2) **Given** ⓢ C **and** C' **tangent to each other at** P.

To prove that the line of centers CC' *passes through* P.

Proof. Let AB be the line tangent to ⓢ C and C' at P. § 235

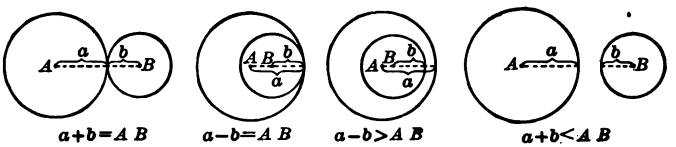
Then, a line through $P \perp AB$ passes through both C and C'. § 253

$\therefore$ P lies on the line of centers.

Q. E. D.

SIGHT WORK

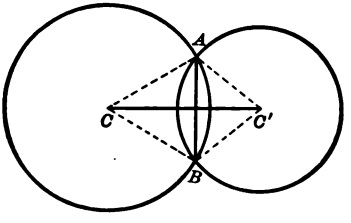
If A and B are the centers of two circles whose radii are a and b, state the various possible relations between the positions of the centers. Which ones are missing among the illustrations here given?



$a+b=AB$ $a-b=AB$ $a-b>AB$ $a+b<AB$

LINE OF CENTERS BISECTS COMMON CHORD

263. THEOREM X. *A line connecting the centers of two intersecting circles is the perpendicular bisector of their common chord.*



Given the intersecting ⊙ C and C' with common chord AB.

To prove that CC' is the ⊥ bisector of AB.

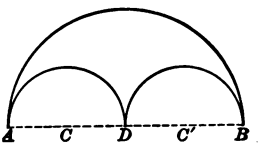
Proof. $CA = CB$ and $C'A = C'B$. Why?

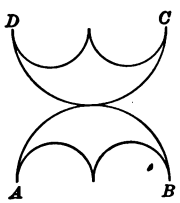
$\therefore$ CC' is the bisector of AB. § 126

(*Two points each equidistant from the extremities, etc.*)

EXERCISES

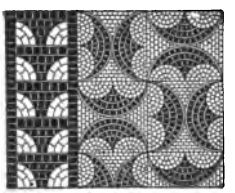
1. In the figure $AD = DB$. Semicircles are constructed on AD, DB, and AB as diameters. Which semicircles are tangent to each other? Prove.





2. In the figure A, B, C, D are the vertices of a square. Show how to construct the entire figure. What semicircles are tangent to each other?

This construction occurs frequently in designs for tile flooring. See accompanying figure. This is from a Roman mosaic.



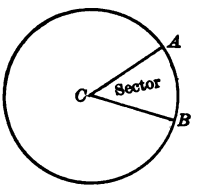
MEASUREMENT OF ANGLES

264. Angles at the Center of a Circle. If the perigon at the center of a circle be divided by radii into 360 equal *angles*, called **degrees** (°), these radii will divide the circle into 360 equal *arcs* according to the theorem, § 244 (1). Hence, we speak of an angle or of an arc of 1°, 2°, 3°, etc. Similarly, a degree may be divided into 60 equal arcs called **minutes** (′).

For this reason a central angle is said to be *measured by the arc which it intercepts.* This means that a given central angle contains a *number of unit angles* equal to the *number of unit arcs* in the intercepted arc.

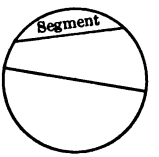
A **quadrant** is an arc of 90°. A **semicircle** is an arc of 180°. A right angle is, therefore, measured by a quadrant and a straight angle is measured by a semicircle.

265. A **sector** is a figure formed by two radii and their intercepted arc.



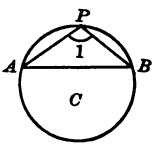
Thus, the sector ACB is formed by the radii BC, AC, and $\overset{\frown}{AB}$.

266. Segment of a Circle. A *segment of a circle*, or a *circle-segment*, is a figure formed by a chord and the arc which it subtends. For each chord there are two circle-segments corresponding to the two arcs which it subtends.



If a chord is a *diameter* the two circle-segments are equal.

267. Angle Inscribed in an Arc or in a Segment. An angle is said to be *inscribed in an arc* if its vertex lies on the arc and its sides meet the arc at its end-points.



Such an angle is also said to be *inscribed in the circle-segment* formed by the arc and its chord.

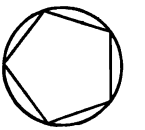
E.g. $\angle 1$ is inscribed in the arc APB or in the segment APB.

The following propositions require no formal proof:

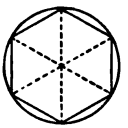
268. *If two angles are equal the arcs which measure them are equal.*

269. *If two angles are measured by equal arcs they are equal.*

SIGHT WORK



1. If a polygon inscribed in a circle is equilateral, how are the intercepted arcs related?

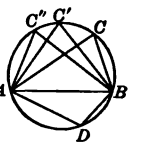


2. If the vertices of an equilateral inscribed polygon are joined to the center of the circle, how are the resulting central angles related?

3. If the polygon of Example 2 has six sides, how many degrees are there in each central angle?

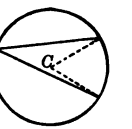
EXPERIMENTAL GEOMETRY

1. Inscribe several angles in the same segment of a circle. Measure the angles and compare their size. State a proposition which seems to be true concerning all angles inscribed in the same segment.



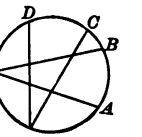
2. If the intersection point C of two lines is made to move along a circle while the lines continue to pass through the fixed points, A and B, on the circle, does the angle C seem to change in size, so long as the point C remains on the same side of the chord AB?

3. Inscribe an angle in each of the two segments into which a chord divides a circle. Measure both angles and find their sum. For example, $\angle C$ and $\angle D$ in the above figure.



4. Inscribe an angle in a circle and then construct a central angle meeting the circle in the same points as the sides of the inscribed angle. Measure the two angles and compare their size.

5. Inscribe an angle in a semicircle and measure it.



6. Inscribe two equal angles in a circle and measure the intercepted arcs. What is a convenient method of deciding whether arcs AB and CD are equal?

MEASURE OF AN INSCRIBED ANGLE

270. THEOREM XI. *An angle inscribed in a circle is measured by one half the intercepted arc.*

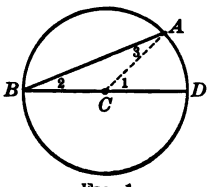


FIG. 1.

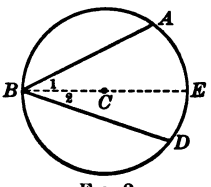


FIG. 2.

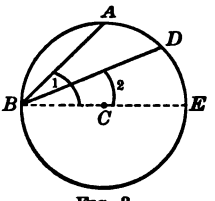


FIG. 3.

Given $\angle ABD$ **inscribed in** $\odot C$ **and intercepting** $\overset{\frown}{AD}$.

To prove that $\angle ABD$ *is measured by* $\frac{1}{2}\overset{\frown}{AD}$.

Case I. When one side, as BD, is a diameter. Fig. 1.

Proof: Draw the radius CA. Then $\angle 1 = \angle 2 + \angle 3$. § 160

(*An exterior* $\angle$ *of a* $\triangle$ *is equal to the sum, etc.*)

Now $CA = CB$ and $\therefore \angle 3 = \angle 2$. Why?

$\therefore \angle 1 =$ twice $\angle 2$, or $\angle 2 = \frac{1}{2}\angle 1$.

Since $\angle 1$ is measured by $\overset{\frown}{AD}$, § 264

$\therefore \angle 2$ is measured by $\frac{1}{2}\overset{\frown}{AD}$.

Case II. When the center C lies within the angle. Fig. 2.

Proof: Draw the diameter BE. Then $\angle ABD = \angle 1 + \angle 2$.

Since $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{AE}$ and $\angle 2$ by $\frac{1}{2}\overset{\frown}{ED}$, Case I

$\therefore \angle ABD$ is measured by $\frac{1}{2}(\overset{\frown}{AE} + \overset{\frown}{ED})$.

That is, $\angle ABD$ is measured by $\frac{1}{2}\overset{\frown}{AD}$.

Case III. When the center lies outside the angle. Fig. 3.

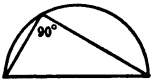
Proof: Draw the diameter BE. Then $\angle ABD = \angle 1 - \angle 2$.

Since $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{AE}$ and $\angle 2$ by $\frac{1}{2}\overset{\frown}{DE}$, Case I

$\therefore \angle ABD$ is measured by $\frac{1}{2}(\overset{\frown}{AE} - \overset{\frown}{DE})$.

That is, $\angle ABD$ is measured by $\frac{1}{2}\overset{\frown}{AD}$. Q. E. D.

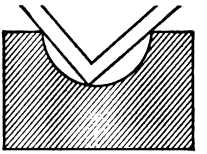
271. COROLLARY 1. *If equal angles are inscribed in the same circle or in equal circles the intercepted arcs are equal.*



272. COROLLARY 2. *An angle inscribed in a semicircle is a right angle.*

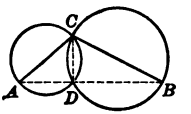
EXERCISES

1. If the vertices of a quadrilateral lie on a circle, either two of its opposite angles are supplementary.



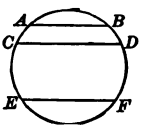
2. In testing core-boxes, pattern makers use a square as indicated in the figure. If the core is a true semicircle will the corner of the square touch it at every point? Why?

3. Two circles intersect at C and D. Diameters CA and CB are drawn. Prove that A, D, B lie on a straight line.



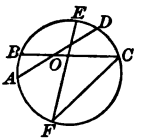
EXPERIMENTAL GEOMETRY

1. Draw several parallel chords in a circle. Compare the two arcs intercepted between any such pair of chords. That is, compare $\overset{\frown}{AC}$ and $\overset{\frown}{BD}$; $\overset{\frown}{CE}$ and $\overset{\frown}{DF}$; $\overset{\frown}{AE}$ and $\overset{\frown}{BF}$. Do this by measuring the chords AC, BD, etc. Formulate a proposition based on these measurements.

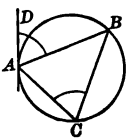


2. Draw two chords AD and BC intersecting at O within a circle. Lay off $\overset{\frown}{DE} = \overset{\frown}{AB}$. Then $\overset{\frown}{CE} = \overset{\frown}{CD} + \overset{\frown}{AB}$. Connect C and E to a point F on the circle. Measure and compare $\angle F$ and $\angle COD$.

By § 270 $\angle F$ is measured by $\frac{1}{2}\ \overset{\frown}{CE}$. Hence, it appears that $\angle COD$ is measured by $\frac{1}{2}\ \overset{\frown}{CE} = \frac{1}{2}(\overset{\frown}{CD} + \overset{\frown}{AB})$.



3. Construct a tangent AD to a circle at one end of a chord AB. Join A and B to any point C on the circle but not on the arc intercepted by $\angle BAD$. Measure and compare $\angle C$ and $\angle BAD$. Complete the following statement: "Since $\angle C$ is measured by $\frac{1}{2}\ \overset{\frown}{AB}$ it follows that $\angle BAD$ is measured by . . . ."



ARCS INTERCEPTED BY PARALLEL LINES

273. THEOREM XII. *Two parallel lines meeting a circle intercept equal arcs on it.*

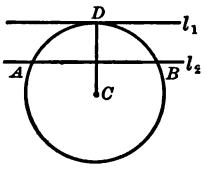


FIG. 1.

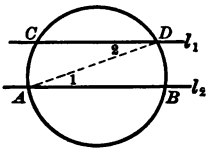


FIG. 2.

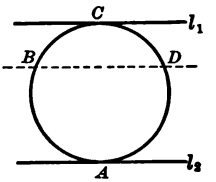


FIG. 3.

Given $\parallel$ **lines** l_1, l_2 **meeting a** $\odot$.

To prove that the intercepted arcs are equal.

Case I. When one line l_1 is a tangent and the other line l_2 is a secant. Fig. 1.

Proof: Draw the radius CD to the point of tangency.

Then $CD \perp l_1$. § 252

(*If a line is tangent to a* $\odot$, *the radius drawn to the point of contact is* $\perp$ *to the tangent.*)

$\therefore CD \perp l_2$. § 151

(*A line* $\perp$ *to one of two* $\parallel$ *lines, etc.*)

$\therefore \overset{\frown}{AD} = \overset{\frown}{BD}$. § 255

(*A diameter* $\perp$ *to a chord, etc.*)

Case II. When l_1 and l_2 are both secants. Fig. 2.

Proof: Draw the chord AD.

Then $\angle 1 = \angle 2$, and therefore $\overset{\frown}{AC} = \overset{\frown}{BD}$. Why?

Case III. When l_1 and l_2 are both tangents. Fig. 3.

Proof: Draw secant $BD \parallel l_1$. $\therefore BD \parallel l_2$.

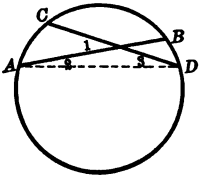
Then $\overset{\frown}{AB} = \overset{\frown}{AD}$ and $\overset{\frown}{BC} = \overset{\frown}{DC}$. Case I.

$\therefore \overset{\frown}{AB} + \overset{\frown}{BC} = \overset{\frown}{AD} + \overset{\frown}{DC}$ or $\overset{\frown}{ABC} = \overset{\frown}{ADC}$. Why?

Q. E. D.

MEASURE OF ANGLE FORMED BY INTERSECTING CHORDS

274. THEOREM XIII. *An angle formed by two intersecting chords is measured by one half the sum of the intercepted arcs.*



Given $\angle 1$ formed by the chords AB and CD.

To prove that measure $\angle 1 = \frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD})$.

Proof: Draw AD.

Then $\angle 1 = \angle 2 + \angle 3$. § 160

$\therefore$ Measure $\angle 1$ = measure $\angle 2$ + measure $\angle 3$. § 268

But measure $\angle 2 = \frac{1}{2}\overset{\frown}{BD}$, and measure $\angle 3 = \frac{1}{2}\overset{\frown}{AC}$. § 270

(*An $\angle$ inscribed in a $\odot$ is measured, etc.*)

$\therefore$ Measure $\angle 1 = \frac{1}{2}\overset{\frown}{AC} + \frac{1}{2}\overset{\frown}{BD} = \frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD})$. Why?

Q. E. D.

SIGHT WORK

1. Find $\angle 1$ in the above figure (*a*) if $\overset{\frown}{BD} = 30°$ and $\overset{\frown}{AC} = 40°$; (*b*) if $\overset{\frown}{BD} = 12°$ and $\overset{\frown}{AC} = 18°$.

2. Find $\angle 1$ in the above figure (*a*) if $\overset{\frown}{CB} = 110°$ and $\overset{\frown}{ACD} = 170°$; (*b*) if $\overset{\frown}{CB} = 150°$ and $\overset{\frown}{ACD} = 200°$.

3. If in the same figure $\angle 1 = 20°$ and $\overset{\frown}{BD} = 30°$ find $\overset{\frown}{AC}$. Also find $\overset{\frown}{AC}$ if $\angle 1 = 50°$ and $\overset{\frown}{BD} = 60°$.

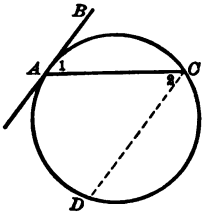
4. If in Fig. 1, p. 138, $\overset{\frown}{AD} = 55°$, find all the other arcs. Also find the other arcs if $\overset{\frown}{BD} = 110°$.

5. If in Fig. 2, p. 138, $\overset{\frown}{AC} = 40°$ and $\overset{\frown}{CD} = 150°$, find $\overset{\frown}{DB}$ and also $\overset{\frown}{AB}$.

6. If in Fig. 3, p. 138, $\overset{\frown}{BC} = 80°$, find $\overset{\frown}{CD}$, $\overset{\frown}{DA}$, and $\overset{\frown}{AB}$. Also find the other arcs if $\overset{\frown}{AD} = 95°$.

MEASURE OF ANGLE FORMED BY TANGENT AND CHORD

275. THEOREM XIV. *An angle formed by a tangent and a chord drawn from the point of tangency is measured by one half the intercepted arc.*



Given $\angle 1$ **formed by tangent** AB **and chord** AC.

To prove that measure $\angle 1 = \frac{1}{2}\overset{\frown}{AC}$.

Proof: Draw chord $CD \parallel AB$.

Then	$\angle 1 = \angle 2$,	Why?
and	measure $\angle 1$ = measure $\angle 2$.	§ 268
Now	$\overset{\frown}{AC} = \overset{\frown}{AD}$.	§ 273

(*Two* $\parallel$ *lines meeting a* $\odot$, *intercept equal arcs on it.*)

But	measure $\angle 2 = \frac{1}{2}\overset{\frown}{AD} = \frac{1}{2}\overset{\frown}{AC}$.	§ 270

(*An* $\angle$ *inscribed in a* $\odot$ *is measured by one half the intercepted arc.*)

$\therefore$ measure $\angle 1 = \frac{1}{2}\overset{\frown}{AC}$. Why?

Q. E. D.

SIGHT WORK

1. Find $\angle 1$ in the above figure (*a*) if $\overset{\frown}{AC} = 70°$; (*b*) if $\overset{\frown}{AC} = 60°$; (*c*) if $\overset{\frown}{AC} = 47°$; (*d*) if $\overset{\frown}{AC} = 35°\ 30'$.

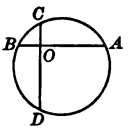
2. How many degrees in $\overset{\frown}{AC}$ (*a*) if $\angle 1 = 40°$; (*b*) if $\angle 1 = 35°$; (*c*) if $\angle 1 = 23°\ 15'$?

3. How many degrees are there in $\angle 1$ (*a*) if $\overset{\frown}{AD} = 70°$; (*b*) if $\overset{\frown}{AD} = 110°$; (*c*) if $\overset{\frown}{AD} = 84°\ 30'$?

4. Find $\angle 1$ (*a*) if $\overset{\frown}{CD} = 80°$; (*b*) if $\overset{\frown}{CD} = 170°\ 20'$.

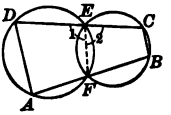
EXERCISES

1. Draw two tangents to a circle and the chord connecting the points of tangency. Prove that two equal angles are formed.



2. The sum of two opposite arcs intercepted by a pair of perpendicular chords equals a semicircle.

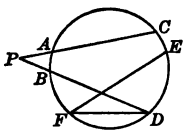
3. Lines are drawn through the points of intersection of two circles and limited by the circles as shown in the figure. Show that the chords joining the corresponding ends of these lines are parallel.



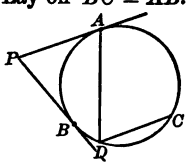
Suggestion. Prove that (*a*) $\angle A + \angle 1 = 2$ rt. $\measuredangle$; (*b*) $\angle A = \angle 2$; (*c*) $\angle 2 + \angle B = 2$ rt. $\measuredangle$; (*d*) $\angle A + \angle B = 2$ rt. $\measuredangle$.

EXPERIMENTAL GEOMETRY

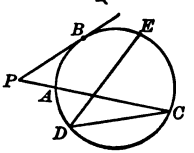
1. Draw two secants meeting outside a circle and intercepting $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$. Lay off $\overset{\frown}{CE} = \overset{\frown}{AB}$. Then $\overset{\frown}{DE} = \overset{\frown}{CD} - \overset{\frown}{AB}$. Connect D and E with a point F on the circle. Measure $\angle P$ and $\angle F$. Are they equal? Since $\angle F$ is measured by $\frac{1}{2}\overset{\frown}{DE}$, how is $\angle P$ measured? Does this seem to show that $\angle P$ is measured by one-half the difference of its intercepted arcs? that is, by $\frac{1}{2}(\overset{\frown}{CD} - \overset{\frown}{AB})$?



2. Draw two tangents to a circle meeting at P. Lay off $\overset{\frown}{BC} = \overset{\frown}{AB}$. Then $\overset{\frown}{AC} = \overset{\frown}{ACB} - \overset{\frown}{AB}$. Connect C and A with a point D on the circle. Measure and compare $\angle P$ and $\angle D$. Does this seem to show that $\angle P$ is measured by one-half the difference between its intercepted arcs, that is, by $\frac{1}{2}(\overset{\frown}{ACB} - \overset{\frown}{AB})$.



3. Draw a tangent and a secant meeting at P. Lay off $\overset{\frown}{BE} = \overset{\frown}{AB}$. Complete the figure and show that $\angle P$ seems to be measured by $\frac{1}{2}(\overset{\frown}{BC} - \overset{\frown}{AB})$.



MEASURE OF ANGLES FORMED BY SECANTS AND TANGENTS

276. THEOREM XV. *The angle formed by two secants, two tangents, or a tangent and a secant, meeting outside a circle, is measured by one half the difference of the intercepted arcs.*

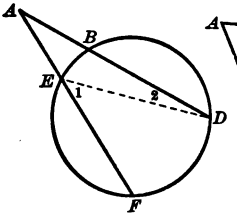


FIG. 1.

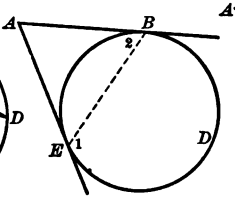


FIG. 2.

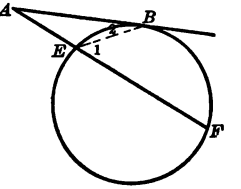


FIG. 3.

Case I. **Given $\angle A$ made by the secants AD and AF.** Fig. 1.

To prove that measure $\angle A = \frac{1}{2}(\overset{\frown}{DF} - \overset{\frown}{BE})$.

Proof: Draw chord ED.

Then $\angle 1 = \angle A + \angle 2$, or $\angle A = \angle 1 - \angle 2$. § 160

Hence, measure $\angle A$ = measure $\angle 1$ − measure $\angle 2$. § 268

But measure $\angle 1 = \frac{1}{2}\overset{\frown}{DF}$ and measure $\angle 2 = \frac{1}{2}\overset{\frown}{BE}$. § 270

(*An angle inscribed in a circle, etc.*)

$\therefore$ measure $\angle A = \frac{1}{2}\overset{\frown}{DF} - \frac{1}{2}\overset{\frown}{BE} = \frac{1}{2}(\overset{\frown}{DF} - \overset{\frown}{BE})$.

Case II. **Given $\angle A$ formed by tangents AB and AE.** Fig. 2.

To prove that measure $\angle A = \frac{1}{2}(\overset{\frown}{BDE} - \overset{\frown}{BE})$.

Proof: Draw chord BE and show that $\angle A = \angle 1 - \angle 2$.

Use measure $\angle 1 = \frac{1}{2}\overset{\frown}{BDE}$ and measure $\angle 2 = \frac{1}{2}\overset{\frown}{BE}$, and proceed as in Case I.

Case III. **Given $\angle A$ made by tangent AB and secant AF.**

To prove that measure $\angle A = \frac{1}{2}(\overset{\frown}{BF} - \overset{\frown}{BE})$.

Proof: Show that $\angle A = \angle 1 - \angle 2$ and proceed as before.

SIGHT WORK

1. If in Fig. 1, page 142, $\overset{\frown}{DF} = 50°$, $\overset{\frown}{BE} = 20°$, find $\angle A$.

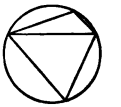
2. If in Fig. 2, page 142, $\overset{\frown}{BDE} = 280°$, how many degrees are there in $\overset{\frown}{BE}$? Find $\angle A$.

3. If in Fig. 3, page 142, $\angle A = 20°$ and $\overset{\frown}{BF} = 100°$, find $\overset{\frown}{BE}$.

4. Compare all angles inscribed in the same circle-segment. If the vertex P is made to move along the arc while the sides pass through the points A and B, will the angle change in size? Prove.

5. If an arc is less than a semicircle, is the inscribed angle greater or less than a right angle?

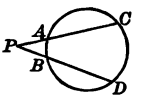
6. If an arc is greater than a semicircle, is the inscribed angle greater or less than a right angle?



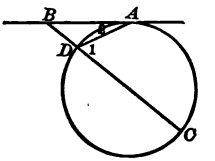
7. Tangents are drawn at the extremities of an arc of 120°. How many degrees are there in the angle formed by these tangents?

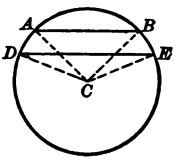
8. Is the following statement true? "The angle formed by two lines meeting a circle is measured by one-half the sum, or one-half the difference, of the intercepted arcs, according as the vertex is inside or outside the circle." What theorems are included in this statement?

9. In the figure suppose the points C and D are fixed while P moves nearer to the circle. What happens to $\overset{\frown}{AB}$? When P reaches the circle what about $\overset{\frown}{AB}$? How may the statement of Example 8 be modified to include the theorem of § 270?



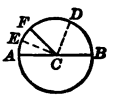
10. If in the figure AB is tangent to the circle and $\angle 1 = 65°$ and $\angle 2 = 25°$, find $\angle B$. Is BC a diameter?





11. In the figure, chords AB and DE are parallel. The center of the circle is joined to the extremities of the chords. Prove that $\angle ACD = \angle BCE$.

12. In the figure, AB is a diameter. If $\overset{\frown}{AE} = \overset{\frown}{EF}$ and $\overset{\frown}{FD} = \overset{\frown}{DB}$ show that $\angle ECD$ is a right angle.



EXERCISES

1. Inscribe any quadrilateral in a circle and verify that the sum of its angles is four right angles.

Suggestion. The measures of $\measuredangle A$, B, C, D, are $\frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{CD})$, $\frac{1}{2}(\overset{\frown}{CD} + \overset{\frown}{DA})$, $\frac{1}{2}(\overset{\frown}{DA} + \overset{\frown}{AB})$, and $\frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{BC})$ respectively. The sum of these measures is the whole circle, and hence the sum of the angles is 360°.

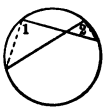
2. Inscribe any pentagon in a circle and verify that the sum of its angles is six right angles.

Suggestion. Show that the sum of the arcs subtended by the angles is three times the circle.

3. Inscribe any hexagon in a circle and verify that the sum of its angles is eight right angles.

4. If the vertices of a trapezoid lie on a circle, its diagonals are equal and the trapezoid is isosceles.

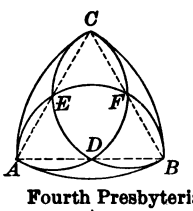
5. Ends of intersecting chords are joined as shown in the figure. Prove that the triangles formed are mutually equiangular.

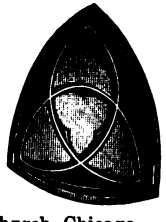


6. With a given segment as a diagonal to construct a square.

Suggestion. Draw a circle on the diagonal as a diameter.

7. On the sides of the equilateral triangle ABC as diameters, semicircles are drawn, such as $AEFB$. Also arcs are drawn with A, B, C as centers and AB as radius, such as $\overset{\frown}{AB}$, $\overset{\frown}{BC}$. Prove that the arcs $AEFB$, $BDEC$, and $CFDA$ meet in pairs at the middle points D, E, F of the sides of the triangle.



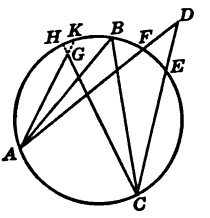


Fourth Presbyterian Church, Chicago.

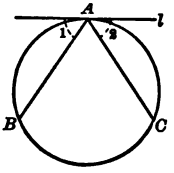
Suggestion. If the middle points of the sides of an equilateral triangle are joined, what kind of triangles are formed?

This figure occurs frequently in church windows and other decorative designs.

8. The angle ABC is inscribed in a circle while $\angle ADC$ has its vertex D outside the circle. Prove that $\angle ABC > \angle ADC$.

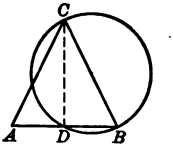


9. In the same figure $\angle AGC$ has its vertex inside the circle. Prove that $\angle AGC > \angle ABC$.

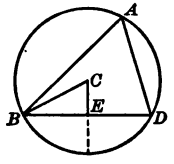


10. The line l is tangent to the circle at the point A. Chords AB and AC are drawn so that $\angle 1 = \angle 2$. Prove that $AB = AC$.

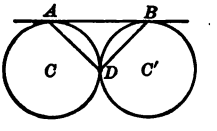
11. A circle is constructed on one side of an isosceles triangle as a diameter. Show that the circle bisects the base.

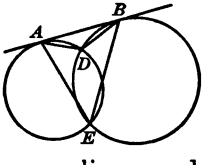


12. A triangle is inscribed in a circle and the center C of the circle is joined to the extremity B of the base. CE is drawn perpendicular to BD. How are the angles BAD and BCE related: (a) if A and C lie on the same side of BD; (b) if A and C lie on opposite sides of BD?



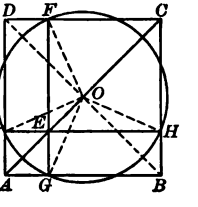
13. Circles C and C' are equal and tangent to each other externally at D. The line AB is tangent to both circles. Prove that ADB is a right angle.





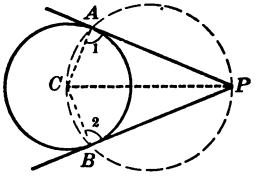
14. Two circles intersect at D and E and have a common tangent AB. Prove that angles ADB and AEB are supplementary.

15. Through any point E in a diagonal of a square lines are drawn parallel to the sides, meeting the sides in F, G, H, and K. If $AO = OC$, show that a circle with center at O and radius OG will pass through the points F, H, and K.



TANGENT TO A CIRCLE FROM A POINT OUTSIDE

277. PROBLEM. *From a given point outside a circle to draw a tangent to the circle.*



Given ⊙ C and an outside point P.

To construct a tangent from P to the circle.

Construction. Draw CP. On CP as a diameter construct a circle, cutting the given circle in the points A and B.

Then PA and PB are both tangent to the circle.

Proof: Draw the radii CA and CB.

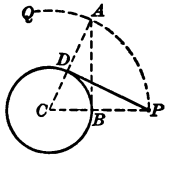
Then $\angle 1 = \angle 2 =$ a rt. $\angle$. § 272

$\therefore$ PA and PB are tangents to ⊙ C. § 251

Q. E. F.

278. COROLLARY. *The two tangents to a circle from a point outside are equal and make equal angles with the line joining the point to the center.*

279. Euclid's Solution of § 277. With C as center and CP as radius draw arc PQ. Draw $BA \perp CP$ meeting $\overset{\frown}{PQ}$ in A. Draw AC and make $PD \perp AC$. Then PD is tangent to the circle.

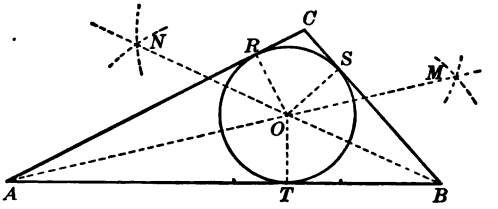


Suggestion. Prove $\triangle PDC = \triangle ABC$ and hence $CD = CB$.

280. Inscribed Circle. A circle is said to be *inscribed in a polygon* if each side of the polygon is tangent to the circle. The polygon is also said to be *circumscribed about the circle.*

INSCRIBING A CIRCLE IN A TRIANGLE

281. PROBLEM. *To inscribe a circle in a given triangle.*



Given a triangle *ABC*.

To inscribe a circle in it.

Construction and Proof. Using § 198, find the point O equally distant from the sides of the triangle.

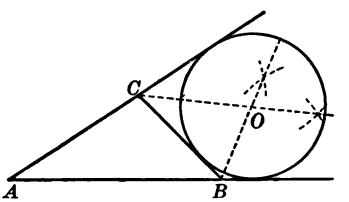
Draw $OR \perp AC$, $OT \perp AB$, and $OS \perp BC$.

Since $OR = OT = OS$, a circle with O as center and radius OR is tangent to the three sides of the triangle. § 251

Q. E. F.

282. **Escribed Circle of a Triangle.** A circle tangent to one side of a triangle and to the other two sides produced is an *escribed circle* of the triangle.

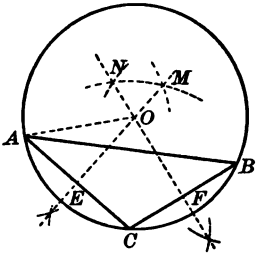
283. PROBLEM. *To construct an escribed circle of a triangle.*



Outline of Solution. Prolong two sides of the triangle forming two exterior angles as shown in the figure. Bisect these angles by lines meeting at O. From O draw perpendiculars to the lines AB, BC, and AC. Prove these perpendiculars equal as in § 281.

CIRCUMSCRIBING A CIRCLE ABOUT A TRIANGLE

284. Problem. *To circumscribe a circle about a given triangle.*



Given the $\triangle ABC$.

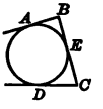
To construct a $\odot$ *passing through its vertices.*

Construction and Proof. Using the method of § 196, find the point O equally distant from A, B, and C.

With O as center and radius OA construct the circle. This is the required circle since $OA = OB = OC$. Q. E. F.

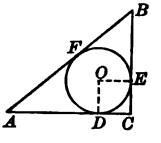
EXERCISES

1. The lines AB, BC, and CD are tangent to a circle at the points A, E, and D respectively. Show that $BC = BA + CD$.

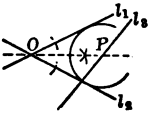


2. Show that in a right triangle the hypotenuse plus the diameter of the inscribed circle is equal to the sum of the two legs of the triangle.

Suggestion. $BE + AD = AB$. $CD + CE =$ diameter.

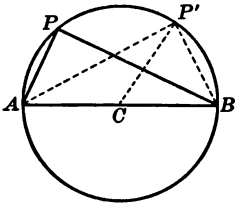


3. Given three straight lines, l_1, l_2, l_3. Construct a circle tangent to l_1 and l_2 and having its center on l_3. Is this construction possible in case the three lines meet in a point?



VERTICES OF ALL RIGHT ▲ ON THE SAME HYPOTENUSE

285. PROBLEM. *To find the locus of the vertices of all right triangles having the same hypotenuse.*



Suppose all possible rt. ▲ to be constructed on the hypotenuse AB.

To find the locus of the vertices P, P', etc.

Construction and Proof: On AB as a diameter construct $\odot C$.

(1) Every point P on $\odot C$ is the vertex of a rt. $\triangle$ with hypotenuse AB. § 272

(2) The vertex P' of every rt. $\triangle$ with hypotenuse AB lies on $\odot C$.

For if P' is inside the $\odot$, then $\angle AP'B >$ rt. $\angle$, § 274

and if P' is outside the $\odot$, then $\angle AP'B <$ rt. $\angle$. § 276

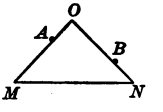
Hence the required locus is the circle on the diameter AB.

Q. E. F.

286. COROLLARY. *The middle point of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.*

SIGHT WORK

1. If a triangular ruler MNO, right-angled at O, is moved about in the plane so that two fixed points, A and B, lie always on the sides MO and NO respectively, what path does the point O trace?

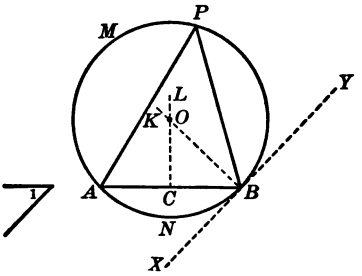


2. If a person walking in a field moves so that the lines from him to two houses always form a right angle, in what kind of a path does he move?

SUPPLEMENTARY PROBLEMS *

SEGMENT OF A CIRCLE CONTAINING A GIVEN ANGLE

287. PROBLEM. *On a given line as a chord to construct a segment of a circle which shall contain a given inscribed angle.*



Given a line AB and an angle such as $\angle 1$.

To construct on AB as a chord a segment of a $\odot$ which shall contain an inscribed $\angle$ equal to $\angle 1$.

Construction and Proof: Through B draw XY so as to make $\angle ABX = \angle 1$.

At B erect $BK \perp XY$, and at C, the middle point of AB, erect $CL \perp AB$.

Then BK and CL meet at O, equidistant from A and B. §§ 154, 191

With O as a center and OB as a radius draw a $\odot$.

Let $\angle P$ be any angle inscribed in the segment AMB.

Then, measure $\angle P = \frac{1}{2}\overset{\frown}{ANB}$ and measure $\angle \overset{\frown}{ABX} = \frac{1}{2}\overset{\frown}{ANB}$.

$\therefore \angle P = \angle ABX = \angle 1$. Q. E. F.

* Pages 150–151 may be omitted without destroying the continuity.

CONSTRUCTING COMMON TANGENTS TO TWO CIRCLES

288. PROBLEM. *To draw a common tangent to two circles neither of which lies wholly inside the other.*

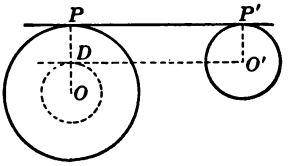


FIG. 1.

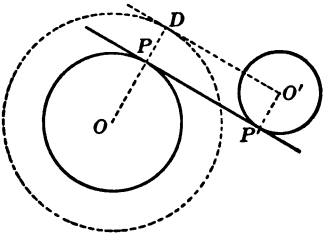


FIG. 2.

Given ⓢ O **and** O' **such that the radius of the first is the greater.**

Required to draw a tangent common to both circles.

Construction. Draw an auxiliary circle with center O and radius equal to the *difference* of the two given radii in figure 1 and equal to the *sum* of these radii in figure 2.

In each case draw a tangent $O'D$ to this auxiliary circle.

Draw the radius OD, thus fixing the point P.

Draw $O'P' \parallel DP$, thus fixing the point P'.

Then PP' is the required tangent.

Outline of Proof: Show that, in each case, $PP'O'D$ is a rectangle, thus making PP' perpendicular to the radii OP and $O'P'$, that is, tangent to each circle.

289. Definition. A common tangent to two circles is called **direct** if it does not cross the line-segment connecting the centers, and **transverse** if it does cross this segment.

Thus, PP' is a direct tangent in Fig. 1, and transverse in Fig. 2.

SIGHT WORK

How many common tangents can be drawn to two circles, (*a*) if each lies entirely outside the other? (*b*) if they are tangent externally? (*c*) if they intersect each other? (*d*) if they are tangent internally? (*e*) if one lies entirely inside the other?

REVIEW EXERCISES INVOLVING LOCI

Describe without proof the plane loci which fulfill the conditions in the following Exercises.

Review §§ 189–191.

1. All points equidistant from the sides of an angle.

2. All points equidistant from two points.

3. All points at a given distance from a given point.

4. All points equidistant from two parallel lines.

5. Centers of all circles tangent to each of two parallel lines.

6. Centers of circles tangent to both sides of an angle.

7. All points equidistant from two intersecting lines. Does this locus include more than one line?

8. Centers of all circles tangent to two intersecting lines.

9. All points at a given distance from a given line.

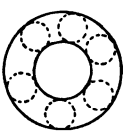
10. Centers of all circles of a given radius tangent to a given line.

11. Centers of all circles through two fixed points.

12. Centers of circles with a fixed radius and passing through a fixed point.

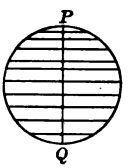
13. Vertices of all isosceles triangles having the same base.

14. Centers of all circles tangent to a line at a fixed point. Is the given point part of the locus?



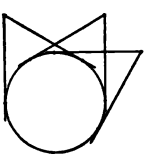
15. Centers of all circles tangent to each of two concentric circles.

16. Vertices of all triangles having a common base and equal altitudes.



17. The middle points of a set of parallel chords. Are the points P and Q in the figure part of the locus?

18. The middle points of all chords of equal length in a given circle.



19. The vertices of triangles having a fixed base and equal medians on that base.

20. An angle of 60° has its sides tangent to a fixed circle. What is the locus of the vertex of the angle?

SOLUTION OF LOCI PROBLEMS

290. General Suggestions.

1. Endeavor to see directly what the required locus is.

2. If necessary construct a number of points which have the given property. This may make the locus apparent.

3. Make sure that all points on the supposed locus have the given property.

Sometimes one or more isolated points on what is supposed to be the locus may not have the given property. Thus, the locus of the vertices of isosceles triangles having a fixed base is the perpendicular bisector of the base, *except* the middle point of the base, for this point is not the vertex of one of the specified triangles.

Examine the locus in § 285 in this respect.

4. Make sure that the supposed locus contains all points having the given property.

Thus, the locus of all points at a given distance from a fixed line includes two lines parallel to it, not *one* alone.

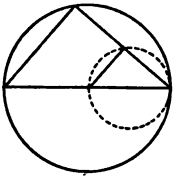
5. Finally, give a formal proof of the two theorems which are necessary to establish a locus; namely,

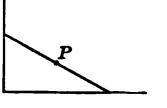
(*a*) All points on the locus have the given property.

(*b*) All points with the given property lie on the locus.

EXERCISES

1. A line of fixed length moves so that its end points touch two fixed lines at right angles to each other. Find the locus of the middle point of the moving line.

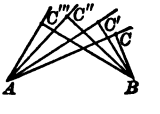




2. Find the locus of the middle points of all chords through a fixed point on a circle.

Use § 255 and then § 285.

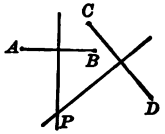
3. Through a fixed point A lines are drawn. Find the locus of the points in which perpendiculars from a fixed point B meet these lines.



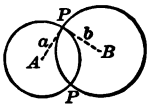
SOLUTION OF LOCI PROBLEMS

291. Intersection of Loci. In general, one condition imposed on a point in a plane restricts it to some figure or locus. If two conditions are imposed, the point is restricted so that it must lie on each of two loci; that is, it must be an intersection point of these loci.

Example 1. To find a point equidistant from the points A and B and also equidistant from C and D, construct perpendicular bisectors of AB and CD. The intersection point P of these perpendiculars is the required point.



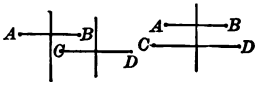
Example 2. To find a point at a distance a from the point A and at a distance b from B, construct a circle with center A and radius a and another with center B and radius b. The intersection points will be the required points.



Example 3. To find a point equidistant from two intersecting lines and at a given distance from a given point, construct the bisectors of the angles formed by the lines; also a circle with the given point as its center and the given distance as its radius. The points at which these bisectors meet the circle are the required points.

292. Discussion of Loci Problems. The discussion of a problem consists in giving all the essentially different cases possible under the conditions specified.

Thus, in Example 1, the perpendicular bisectors of AB and CD will meet unless AB and CD are parallel. In that case there is in general no point P satisfying the two requirements. If, however, AB and CD are so related that the perpendicular bisector of AB is also the perpendicular bisector of CD, then all points on this bisector satisfy the two requirements.



293. Methods of Attacking Problems. Methods of attacking problems were explained in §§ 212–219. We may now add a most important method, the *intersection of loci*, as explained above and further illustrated in the exercises on the following page.

SIGHT WORK

Give the method for finding the points required in each of the following. Discuss each. Give no proofs.

1. A point 2 inches from P and 3 inches from Q.

2. A point 4 inches from P and equidistant from Q and R.

3. A point at a distance a from P and equidistant from the sides of an angle B.

4. A point equidistant from P and Q and equidistant from two intersecting lines.

5. A point equidistant from P and Q and equidistant from two parallel lines.

6. A point equidistant from lines l_1 and l_2 and also from l_3 and l_4. Give all cases according as $l_1 \parallel l_2$ or not and $l_3 \parallel l_4$ or not.

7. A point at a fixed distance from each of two lines.

8. A point at a fixed distance from a line and equidistant from two points.

9. A point at a fixed distance from a line and equidistant from two other lines.

EXERCISES

1. Construct a circle with its center in one side of a triangle and tangent to the other two sides (extended if necessary).

2. Find a point equidistant from two concentric circles, measured along a common radius, and also equidistant from two given points.

3. Construct a circle passing through two given points and having its center on a given line.

4. Find the center of a circle with radius r tangent to a given line l, the center being on a given line l_1.

5. Construct a circle having its center on a given circle and tangent to two given intersecting lines.

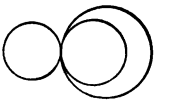
6. Construct a circle having its center on a given line and tangent to two given parallel lines.

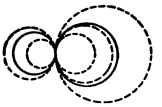
7. Construct a circle with a given radius and tangent to two given intersecting lines.

GENERAL PROBLEMS ON LOCI

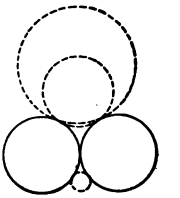
1. Find the locus of the centers of all circles tangent to a fixed circle at a fixed point P. Is the fixed point P a part of this locus? Is the center of the fixed circle a part of it?

2. Two circles are tangent to each other *internally*. Find the locus of the centers of all circles tangent to both *externally*.

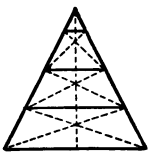




3. Two circles are tangent to each other *externally*. Find the locus of the centers of all circles tangent to both, but *external* to one and *internal* to the other.



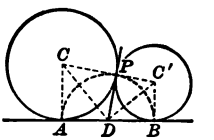
4. Two equal circles are tangent to each other externally. Find the locus of the centers of all circles tangent to both. Is the conclusion altered if the circles intersect? If they do not meet?



5. Find the locus of the points of intersection of the diagonals of trapezoids formed by the sides of an isosceles triangle and lines parallel to its base.

6. All possible parallelograms are constructed with a fixed line as base. Find the locus of the intersection points of the bisectors of its base angles.

7. Find the locus of the points of contact of two varying circles tangent to each other, and each tangent to a given line at a given point.



Suggestion. A and B are the fixed points, and P the point of contact of the circles. Draw the common tangent PD. Prove $AD = DP$ and $DB = DP$.

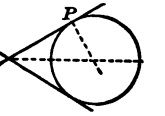
Hence, D is the middle point of AB and DP is constant. That is, the locus is a circle of which AB is a diameter.

8. Find the locus of the middle points of all segments connecting two parallel lines.

FURTHER CONSTRUCTIONS INVOLVING CIRCLES

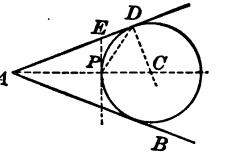
294. Constructions. Besides the constructions already considered the following examples show the most important types that are based on the theorems of Chapter II.

Example 1. Given two intersecting lines and a point P on one of them, to construct a circle tangent to the two lines and passing through P.



Analysis. The center of the required circle must lie on the perpendicular to one of the given lines at P and also on the bisector of an angle formed by the lines. Why? Is there more than one solution?

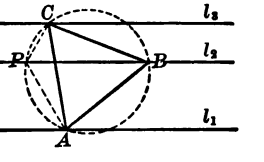
Example 2. Through a point on the bisector of an angle to construct a circle tangent to both sides of the angle.



Construction. Through the given point P draw $EP \perp$ to AP. Lay off $ED = EP$ and at D construct $DC \perp AD$ meeting the line AP in C. Then C is the center of the required circle.

Prove $CP = CD$ and discuss.

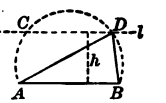
Example 3. Given three parallel lines, to construct an equilateral triangle whose vertices shall lie on these lines.



Solution. Let P be any point on the middle line l_2. Draw PC and PA, each at an angle of 60° with l_2. Through the points A, P, C construct a circle meeting l_2 in B. Then ABC is the required triangle.

Suggestion. Compare $\measuredangle BPA$ and BCA also $\measuredangle BPC$ and BAC.

Example 4. On a given base construct a triangle having a given altitude and a given vertex angle.

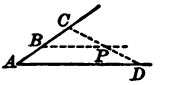


Suggestions. Let AB be the given base. Draw a line $l \parallel AB$ with distance from AB equal to the given altitude h. As in § 287 construct on AB as a chord a segment of a circle which shall contain an inscribed angle equal to the given vertex angle. Let the arc so constructed meet the line l in C and D. Then $\triangle ADB$ is the required triangle.

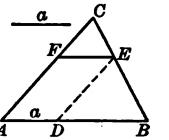
Omit Example 4 if § 287 was omitted.

EXERCISES IN CONSTRUCTION

1. Through a point P within an angle draw a segment terminated by the sides of the angle which shall be bisected by P.



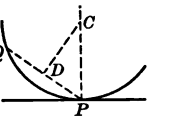
2. Construct a segment, equal to a given segment a, parallel to the base of a triangle and terminated by the other two sides of the triangle.



3. Through a given point within a circle construct a chord equal in length to a given chord.

4. Construct a rectangle having given one side and an angle between the diagonals.

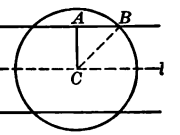
5. Construct a circle tangent to a given line at a given point and passing through another given point.



6. Construct an equilateral triangle, given the radius of the circumscribed circle.

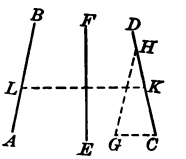
7. Construct an equilateral triangle, given the radius of the inscribed circle.

8. Construct a circle cutting off equal chords of given length on each of two parallel lines.



Suggestion. The center of the circle must be on a line l parallel to the given lines and midway between them. Draw $CA \perp l$ and lay off $AB = \frac{1}{2}$ the given segment.

9. Two non-intersecting line segments AB and CD are given. Without extending AB and CD construct a line EF which would be the bisector of the angle formed by AB and CD if they were extended to meet.

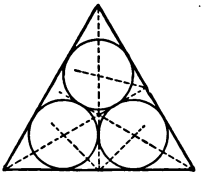


Suggestion. Through a point H on CD draw a line HG, parallel to AB, and construct the isosceles triangle HCG. Draw $KL \parallel CG$. Then EF is the perpendicular bisector of KL. Prove.

EXERCISES IN CONSTRUCTION

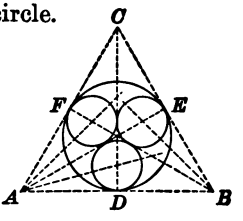
Copy the figures on this page and prove *informally* that the constructions are correct.

1. In an equilateral triangle construct three equal circles, each tangent to the two other circles and to two sides of the triangle.



Suggestion. Construct the altitudes of the triangle and bisect angles as shown in the figure.

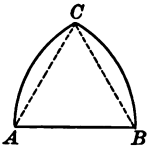
2. Within a given circle construct three equal circles, each tangent to the other two and to the given circle.



Suggestion. Trisect the circle at D, E, and F by making angles at the center each equal to 120°. Draw tangents at D, E, and F, and prove that $\triangle ABC$ is equilateral.

Construct the altitudes and bisect angles as shown in the figure and prove that the centers of the required circles are thus obtained.

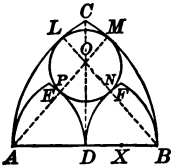
The figure represents a so-called equilateral gothic arch. The arcs AC and BC are drawn from B and A as centers respectively, and with AB as a radius.



The segment AB is called the *span of the arch*, and the point C its *apex*.

3. In the figure $AD = DB$. ABC, ADE, and DBF are equilateral gothic arches.

Construct the circle with center O tangent to the four arcs as shown.

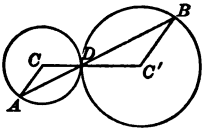


Suggestion. Take X so that $DX = XB$. With centers A and B and radius AX draw arcs meeting at O.

DOOR, UNION PARK CHURCH, CHICAGO.

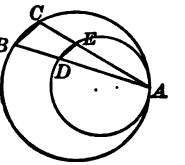
EXERCISES INVOLVING MEASUREMENT OF ANGLES

1. Two circles C and C' are tangent at the point D. AB is drawn through D and terminated by the circles. Prove $CA \parallel C'B$.

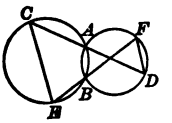


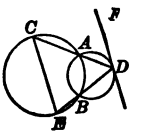
Suggestion. Draw the common tangent at D.

2. Two circles are tangent internally at a point A. Chords AB and AC of the larger circle are drawn meeting the smaller circle in D and E respectively. Prove $BC \parallel DE$.



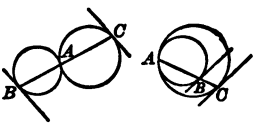
3. Using the first figure, show that $CE \parallel FD$. In the second figure, show that the tangent at D is parallel to CE.





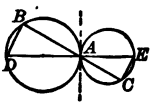
Suggestion. In each case draw the chord AB.

4. Two circles are tangent at A. A secant through A meets the circles at B and C respectively. Prove that the tangents at B and C are parallel.



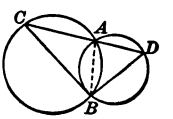
5. Prove that the segments joining one vertex of a regular polygon of n sides to the remaining vertices divide the angle at that vertex into $n - 2$ equal parts.

6. Two circles are tangent externally at A. Lines are drawn through A terminated by the circles. Show that the lines joining the end points of these segments are parallel.



7. Two circles intersect in points A and B. A line CD is drawn through A terminated by the circles. Show that $\angle CBD$ is the same for all positions of CD.

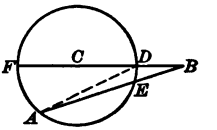
Analysis. We must show that $\overset{\frown}{CA} + \overset{\frown}{AD}$ has a constant number of degrees. Since the two arcs AB are fixed this involves proving that $\overset{\frown}{CB} + \overset{\frown}{DB}$ has a constant number of degrees. How many degrees in the measure of $\angle CAD = \angle CAB + \angle DAB$?



MISCELLANEOUS EXERCISES

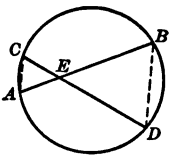
1. If a triangle whose angles are 48°, 56°, and 76° is circumscribed about a circle, find the number of degrees in the arcs into which the points of tangency divide the circle.

2. A triangle ABC whose angles are 45°, 80°, and 55° is inscribed in a circle. Find the angles of the triangle formed by the tangents at A, B, and C.

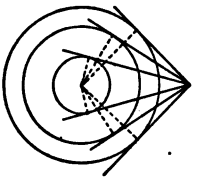


3. In the first figure B is a point in the diameter FD produced. Prove that $BF > BA$ and $BE > BD$.

4. If in the second figure $CE = AE$, prove that $CD = AB$.

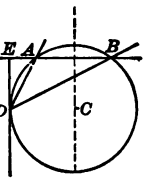


5. Find the locus of points of contact of tangents from a fixed point to a set of concentric circles.

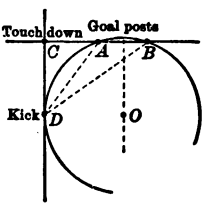


6. Through two given points A and B construct a circle tangent to a given line ED which is perpendicular to the line AB.

Is this construction possible if the given line passes through either of the points A or B? If it meets AB between these points?



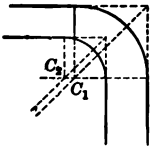
7. In kicking a goal after a touchdown in the game of football, the ball is brought back into the field at right angles to the line marking the end of the field. The distance between the goal posts being given, and also the point at which the touchdown is made, find by a geometrical construction how far back into the field the ball must be brought in order that the goal posts may subtend the greatest possible angle.



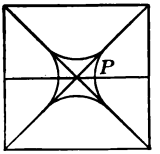
Is the largest angle subtended from the point D as shown in the figure? Why?

APPLICATIONS IN ARTS AND INDUSTRIES

Straight Lines Tangent to Circles

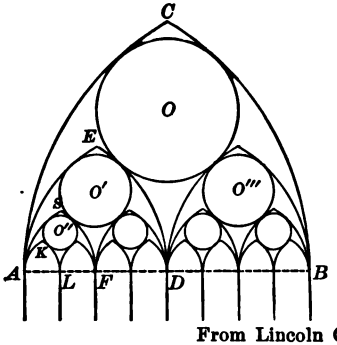


Curved road joining two roads of unequal width.



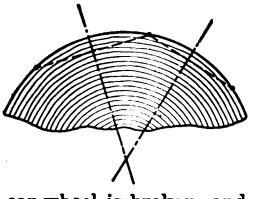
Figures from steel ceiling. Form determined by location of point P.

Circles Tangent to Each Other

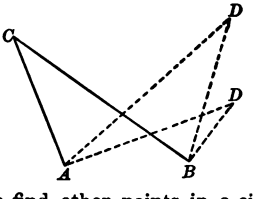


From Lincoln Cathedral, England.

Circles Through Three Points



A car wheel is broken, and it is required to determine its diameter by construction when only a fragment of it is given.



To find other points in a circle which passes through A, B, C. Measure $\angle C$ and find points D such that $\angle D = \angle C$.

René Descartes (1596–1650) was born near Tours in France, and died in Stockholm. On leaving school he went to Paris and gave two years to the study of mathematics. After spending some time in the army and in travel he finally settled in Paris and devoted himself to philosophy and mathematics.

In the year 1629 Descartes moved to Holland, in order to pursue his studies without interruption. From that time he lived in seclusion, carrying on his correspondence with learned men through one or two trusted friends who kept his exact whereabouts a profound secret.

Descartes may be regarded as the first of the modern mathematicians. His greatest work was in connection with geometry, and his contributions in this line have never been surpassed in importance.

BOOK III

MEASUREMENT. PROPORTION. SIMILAR FIGURES

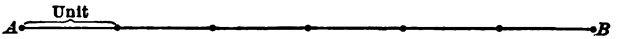
295. Ideal Character of Geometry. The purely ideal character of the elements of geometry, such as *point*, *line*, and *plane*, is evident from the properties ascribed to them. Thus, a point which has position but no length, width, or thickness cannot have a material existence.

The method of treatment is likewise ideal and the results obtained are *exact* propositions which can be *verified only approximately by construction and measurement.* See § 63.

This ideal character of geometry must be borne in mind in order to understand the treatment of measurement.

IDEAL MEASUREMENT

296. Exact Multiples of the Unit. To measure a line-segment, such as AB, a certain segment is selected as a unit, and this is applied successively along the given segment beginning at one end as A. If finally the end point of the unit exactly coincides with B, then the unit is contained in AB an *integral* number of times and AB is an *exact* multiple of the unit.

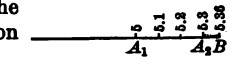


Numerical Measure. The *numerical measure* or simply the *measure* of a geometric quantity is the number of times it contains the unit of measure.

Thus, 6 is the numerical measure, or the **length** of the segment AB, if the unit of length can be laid off on it 6 times in succession.

297. Multiples of Fractional Parts of the Unit. In selecting a unit of measure it may happen that it is not contained an *integral* number of times in the segment to be measured.

Thus, in measuring a line-segment the *meter* is often a convenient unit. Suppose it has been applied five times to the segment AB and at the fifth time the end falls on A_1, A_1B being less than one meter.



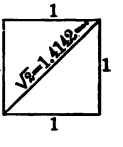
Then, taking a decimeter (one tenth of a meter) as a new unit, suppose this is contained three times in A_1B with a remainder A_2B, less than a decimeter.

Finally, using as a unit a *centimeter* (one tenth of a decimeter), suppose this is contained exactly six times in A_2B.

Then, the length of AB is five meters, 3 decimeters, and 6 centimeters, or 5.36 meters.

298. Segment Incommensurable with the Unit. It may also happen that, in continuing this ideal process of measuring, no fractional part of the unit can be found which exactly measures the given segment, that is, such that the final division point falls *exactly* on B.

E.g. it is known that in a square whose sides are each one unit the diagonal is $\sqrt{2}$, and that this cannot be expressed *exactly* as an integer or as a fraction whose numerator and denominator are both integers.



By the ordinary process of extracting square root we find $\sqrt{2} = 1.4142\cdots$, a process which never terminates.

Hence, in attempting to measure the diagonal of a square whose side is one meter, we find, as an approximation, 1 meter, 4 decimeters, 1 centimeter, 4 millimeters, or 1.414 meters, with a segment less than a millimeter still left unmeasured.

299. Commensurable and Incommensurable Segments. Two straight line-segments are *commensurable* if they have a common unit of measure. Otherwise they are *incommensurable.*

E.g. two line-segments whose lengths are *exactly* 5.27 and 3.42 meters respectively have one centimeter as a common unit of measure, it being contained 527 times in the first segment and 342 times in the second.

But the side and the diagonal of a square have *no common unit of measure.*

PRACTICAL MEASUREMENT

300. Exact Measurement Impossible in Practice. If two different measurements are made of the same distance they will usually fail to agree *exactly*.

Thus, if the distance to be measured is approximately one thousand feet, the measurements may agree within *one foot*, but unless the work is very carefully done, they are not likely to agree within *one inch*, and much less likely to agree within $\frac{1}{16}$ *of an inch*.

Hence *all* practical measurements are **approximations**, the closeness of the approximation depending upon the instruments used and the care with which the work is done.

This is always understood in reading numbers representing measurements. Thus, if the distance between two railway stations is given as 43.7 miles, this is understood to mean that the distance is greater than 43.65 miles and less than 43.75 miles. That is, 43.7 is the nearest approximation in tenths of a mile.

Again, if the length of the meter is given as 39.37 in., this means that its length lies between 39.365 in. and 39.375 in. That is, 39.37 is the closest approximation in hundredths of an inch.

301. Practical Measures Always Commensurable. The measures of quantities obtained by practical measurement are *always commensurable. This is true for the most exact measurement that has ever been made or ever will be made. Hence for practical purposes all quantities of the same kind may be regarded as commensurable.*

The full significance of this statement should be clearly understood. Thus, theorems proved for commensurable quantities are sufficient for *all* practical applications.

The existence of incommensurable quantities is proved by purely logical considerations, and such quantities are discussed only for the sake of logical completeness.

The theorems of this book are proved for commensurable quantities only, the incommensurable cases being treated in the Appendix.

RATIOS OF GEOMETRIC QUANTITIES

302. The **ratio of two commensurable quantities** is the quotient of their numerical measures taken with respect to the same unit.

E.g. if two segments are respectively 3 ft. and 4 ft. in length, the ratio of the first segment to the second is $\frac{3}{4}$.

The ratio of two commensurable quantities is the *same*, no matter what common unit of measure is used.

E.g. two segments whose numerical measures are 3 and 4 if one *foot* is the common unit, have 36 and 48 as their numerical measures if one *inch* is the unit. *But the ratio is the same in both cases;* namely, $\frac{36}{48} = \frac{3}{4}$.

303. The **approximate ratio of two incommensurable quantities** is the quotient of their approximate numerical measures. It will be seen that this approximate ratio depends upon the size of the smallest measuring unit available, and that the approximation can be made as close as we please by taking the measuring unit small enough.

E.g. an approximate ratio of the side of a square to its diagonal is $\frac{1}{1.41} = \frac{100}{141}$. Another and closer approximation is $\frac{1}{1.414} = \frac{1000}{1414}$. In this case the numerical measure of one of the segments is *exact*.

An approximate ratio of $\sqrt{2}$ to $\sqrt{3}$, in which neither has an exact measure, is $\frac{1.41}{1.73} = \frac{141}{173}$. Another approximate ratio is $\frac{1.414}{1.732} = \frac{1414}{1732}$.

304. **The numerical measure** of a quantity is a *number*, as is also the *ratio* of two such measures. It is understood, however, that such measures and ratios are all positive numbers.

RATIO AND PROPORTION

305. **Ratio.** A fraction is often called a *ratio*. Thus $\frac{a}{b}$ may be read *the ratio of a to b*, and is often written $a : b$.

306. **Terms of a Ratio.** The numerator is called the *antecedent* of the ratio, and the denominator the *consequent*. The antecedent and consequent are called the *terms* of the ratio.

307. Proportion. An equation, each of whose members is a ratio, is called a *proportion.*

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion. A proportion is also written

$$a : b = c : d, \text{ or } a : b :: c : d.$$

It is read *the ratio of a to b equals the ratio of c to d*, or briefly, *a is to b as c is to d.*

308. Proportional Numbers. The four numbers a, b, c, and d, when written $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$, are said to be *in proportion* or *to be proportional.*

Means and Extremes. In the proportion $a : b = c : d$ the two end terms, a and d, are called the *extremes*, and the two middle terms, b and c, the *means.*

309. Third Proportional. Mean Proportional. If $a : b = b : x$, then x is called the third proportional to a and b, and b is a *mean proportional* between a and x.

310. Fourth Proportional. If $a : b = c : x$, then x is called the *fourth proportional* to a, b, and c.

311. Continued Proportion. If $a : b = b : c = c : d = d : e$, then a, b, c, d, e are said to be in *continued proportion.*

HISTORICAL NOTE

Theory of Proportion. The simple theory of proportion is of early origin. Thus, it is believed that Pythagoras (560 B.C.) used it in proving the famous theorem known by his name. (See § 350.) Pythagoras also discovered the incommensurable number. But it remained for Eudoxus, the teacher of Plato, to perfect a rigorous theory of proportion.

However, the Greek theory of proportion was exceedingly complicated. Euclid, following his predecessors, dealt with ratios of *magnitudes* in general as well as of *numbers.* Later writers have frequently insisted that ratios in general are not numbers. But nothing is gained by this procedure, since they possess all the properties of numbers. In this book a ratio is treated simply as the quotient of two numbers.

THEOREMS ON PROPORTION

312. THEOREM. *If we have given* $\frac{a}{b}=\frac{c}{d}$,

Then (1) $ad=bc$.

(2) $\frac{b}{a}=\frac{d}{c}$.

(3) $\frac{a}{c}=\frac{b}{d}$.

(4) $\frac{a+b}{b}=\frac{c+d}{c}$.

(5) $\frac{a-b}{b}=\frac{c-d}{c}$.

(6) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

(7) $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

Proof: (1) Multiplying both members of the equation $\frac{a}{b}=\frac{c}{d}$ by bd, we get

$$ad=bc. \qquad \text{Ax. 3, § 71}$$

That is, *if four numbers are in proportion, the product of the means equals the product of the extremes.*

(2) Dividing both members of the equation $bc=ad$ by ac, we get

$$\frac{b}{a}=\frac{d}{c}. \qquad \text{Ax. 4, § 71}$$

This process is called taking the proportion by **inversion.**

(3) Dividing both members of the equation $ad=bc$ by cd, we get

$$\frac{a}{c}=\frac{b}{d}. \qquad \text{Ax. 4, § 71}$$

This process is called taking the proportion by **alternation.**

(4) Adding 1 to each member of the equation $\frac{a}{b}=\frac{c}{d}$, we have

$$\frac{a}{b}+1=\frac{c}{d}+1 \text{ or } \frac{a+b}{b}=\frac{c+d}{d}. \qquad \text{Ax. 1, § 71}$$

This process is called taking the proportion by **composition.**

(5) Subtracting 1 from each member of the equation $\frac{a}{b}=\frac{c}{d}$, we have

$$\frac{a}{b}-1=\frac{c}{d}-1 \text{ or } \frac{a-b}{b}=\frac{c-d}{d}. \qquad \text{Ax. 2, § 71}$$

This process is called taking the proportion by **division.**

(6) Dividing the members of equation (4) by those of (5) we have: $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. Ax. 4, § 71

This process is called taking the proportion by **composition and division.**

(7) Multiplying each member of the equation $\frac{a}{b}=\frac{c}{d}$ by itself, we have $\frac{a^2}{b^2}=\frac{c^2}{d^2}$, $\frac{a^3}{b^3}=\frac{c^3}{d^3}$, $\frac{a^4}{b^4}=\frac{c^4}{d^4}$, etc. Ax. 3, § 71

That is, *if all terms of a proportion are raised to the same power, the results form a proportion.* Q. E. D.

313. COROLLARY 1. *If the product of two numbers is equal to the product of two other numbers, then the first two are the extremes of a proportion of which the other two are the means.*

Proof: If $ad = bc$, then dividing both members by bd we have

$$\frac{a}{b}=\frac{c}{d} \text{ or } a:b=c:d.$$

314. COROLLARY 2. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a+b}{a}=\frac{c+d}{c}$ *and also* $\frac{a-b}{a}=\frac{c-d}{c}$.

Proof: First take the given proportion by inversion, then proceed as in (4) and (5) above.

315. COROLLARY 3. *If in a proportion the antecedents are equal, then the consequents are equal and conversely.*

316. COROLLARY 4. *The mean proportional between two numbers is the square root of their product.*

EXERCISES

1. From $a:b=c:d$ and $m:n=r:s$, show that $am:bn=cr:ds$.

2. If $a:b=c:d$, show that $ma-b:mc-d=a:c$.

3. If $a:b=c:d$, show that $c+d:a+b=d:b$.

FURTHER THEOREMS ON PROPORTION

317. THEOREM. *If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, then the fourth terms are equal.*

Given $\frac{a'}{b}=\frac{c}{d}$ and $\frac{a''}{b}=\frac{c}{d}$.

To prove that $a'=a''$.

Proof: $a'=\frac{bc}{d}$ and $a''=\frac{bc}{d}$. Why?

$\therefore a'=a''$. Why?

318. THEOREM. *In a series of equal ratios the sum of two or more antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

To prove that $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

Proof: Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=r$.

Then $a=br,\ c=dr,\ e=fr$. Why?

Adding, $a+c+e=br+dr+fr=r(b+d+f)$.

Dividing, $\frac{a+c+e}{b+d+f}=r$. But $r=\frac{a}{b}$.

$\therefore \frac{a+c+e}{b+d+f}=\frac{a}{b}$. Why?

319. **Nature of the Quantities in a Proportion.** The terms of a ratio may represent geometric magnitudes, but in such cases they are the *measures* of these magnitudes and are therefore ordinary numbers. Hence all the ordinary operations of arithmetic may be performed on them. A proportion is, therefore, treated in all respects like any ordinary equation, though some of its symbols may represent the measures of lengths, others of areas, etc.

SIGHT WORK

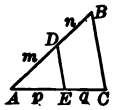
1. Find x in the proportion $x : 4 = 7 : 8$.
2. Find x in the proportion $8 : x = 2 : 8$.
3. Find x in the proportion $4 : 5 = x : 10$.

EXERCISES

1. Find the fourth proportional to 17, 19, and 187.
2. Find a mean proportional between 6 and 54.
3. Find the third proportional to 27 and 189.
4. Find the unknown term in each of the following proportions:

$x : 42 :: 27 : 126$; $78 : x :: 13 : 3$; $99 : 117 :: x : 39$; $171 : 27 :: 57 : x$.

320. **Segments Divided Proportionally.** If D and E are points on the sides of the triangle ABC, and if m, n, p, and q, the numerical measures respectively of AD, DB, AE, and EC, are such that $\frac{m}{n} = \frac{p}{q}$, then the points D and E are said to divide the sides AB and AC *proportionally*, that is, *in the same ratio.* For convenience it is common to let AD, DB, AE, and EC stand for the *numerical measures* of these segments, and thus to write the above proportion,



$$\frac{AD}{DB} = \frac{AE}{EC} \text{ or } AD : DB = AE : EC.$$

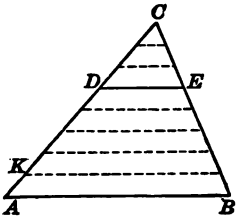
EXPERIMENTAL GEOMETRY

1. Construct a triangle and draw a line parallel to its base cutting the other two sides. Measure to see whether this line divides these sides proportionally. Try this with several triangles of different shapes.

2. Draw three parallel lines and two transversals. Measure the intercepted segments to see whether they form a proportion. That is, is $\frac{AB}{BC}$ equal to $\frac{DE}{EF}$?

PROPORTIONAL DIVISION OF SIDES OF A TRIANGLE

321. THEOREM I. *If a line is parallel to one side of a triangle and cuts the other two sides, then it divides these sides proportionally.*



Given $\triangle ABC$ **in which** $DE \parallel AB$.

To prove that $\dfrac{CD}{DA} = \dfrac{CE}{EB}$.

Proof: Suppose that AK, a common measure* of CD and DA, is contained 3 times in CD and 5 times in DA.

Then $\dfrac{CD}{DA} = \dfrac{3}{5}$. (1)

Through the points of division on CD and DA draw lines parallel to AB, cutting CE and EB. These lines divide CB into 8 equal parts, of which 3 lie on CE and 5 on EB. § 181

(*If* $\parallel$ *lines intercept equal segments on one transversal, etc.*)

Hence $\dfrac{CE}{EB} = \dfrac{3}{5}$. (2)

From (1) and (2), $\dfrac{CD}{DA} = \dfrac{CE}{EB}$. Why?

Q. E. D.

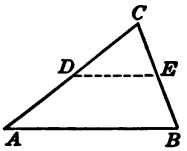
322. Corresponding Segments. If a line cuts two sides of a triangle, the segments formed correspond in certain ways.

Thus in the figure of § 323:

(1) CD and DA correspond to CE and EB.
(2) CA and DA correspond to CB and EB.
(3) CD and CE correspond to DA and EB.
(4) CA and CD correspond to CB and CE.
(5) CA and CB correspond to DA and EB.
(6) CA and CB correspond to CD and CE.

* For the case when CD and DA have no common measure, see § 526 in the Appendix.

323. THEOREM II. *If a line is parallel to one side of a triangle and cuts the other two sides, then any two pairs of corresponding segments form a proportion.*



Given $\triangle ABC$ **with** $DE \parallel AB$.

To prove that (1) $\frac{CD}{DA}=\frac{CE}{EB}$, (2) $\frac{CA}{DA}=\frac{CB}{EB}$, (3) $\frac{CD}{CE}=\frac{DA}{EB}$,

(4) $\frac{CA}{CD}=\frac{CB}{CE}$, (5) $\frac{CA}{CB}=\frac{DA}{EB}$, (6) $\frac{CA}{CB}=\frac{CD}{CE}$.

Proof: (1) $CD : DA :: CE : EB$. § 321

(2) From (1) we have

$$\frac{CD+DA}{DA}=\frac{CE+EB}{EB} \text{ or } \frac{CA}{DA}=\frac{CB}{EB}.$$ § 312, (4)

(3) From (1) we have $\frac{CD}{CE}=\frac{DA}{EB}$. § 312, (3)

(4) From (1) we have

$$\frac{CD+DA}{CD}=\frac{CE+EB}{CE}, \text{ or } \frac{CA}{CD}=\frac{CB}{CE}.$$ § 314

(5) From (2) we have $\frac{CA}{CB}=\frac{DA}{EB}$. Why?

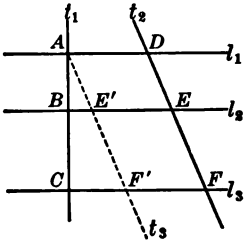
(6) From (4), $\frac{CA}{CB}=\frac{CD}{CE}$. Why?

Q. E. D.

324. COROLLARY. *If a line cuts two sides of a triangle so as to make two pairs of corresponding segments proportional, then any other two pairs of corresponding segments form a proportion.*

SEGMENTS INTERCEPTED BY PARALLEL LINES

325. THEOREM III. *Three or more parallel lines intercept proportional segments on any two transversals.*



Given parallel lines l_1, l_2, l_3 cut by the transversals t_1 and t_2.

To prove that $\dfrac{AB}{BC}=\dfrac{DE}{EF}$.

Proof: Through A draw a transversal $t_3 \parallel t_2$.

Then $AE' = DE$ and $E'F' = EF$. Why?

But $\dfrac{AB}{BC}=\dfrac{AE'}{E'F'}$. § 321

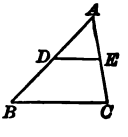
(*If a line is* $\parallel$ *to one side of a* $\triangle$ *and cuts the other two sides, etc.*)

$\therefore \dfrac{AB}{BC}=\dfrac{DE}{EF}$. Why?

Q. E. D.

EXERCISE

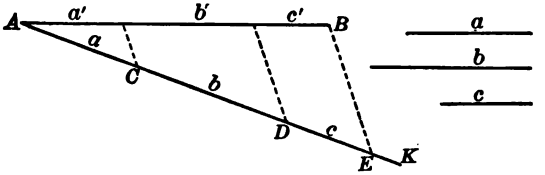
If in $\triangle ABC$ $DE \parallel BC$, compute the segments left blank from those given in the following table:



AD	DB	AE	EC	AB	AC
20	24	15			
4	56		42		
	102	12	408		
25		18	342		

DIVIDING A SEGMENT PROPORTIONALLY TO GIVEN SEGMENTS

326. PROBLEM. *To divide a given line-segment into parts proportional to given segments.*



Given segment AB and segments a, b, c.

To divide AB into parts proportional to a, b, c.

Construction. Through A draw any convenient line AK.

On AK lay off segments $AC = a$, $CD = b$, $DE = c$. Draw EB, and through D and C draw lines parallel to EB, thus dividing AB into the parts a', b', c'.

Then $\dfrac{a'}{b'} = \dfrac{a}{b}$ and $\dfrac{b'}{c'} = \dfrac{b}{c}$. §§ 323, 325

$\therefore \dfrac{a'}{a} = \dfrac{b'}{b} = \dfrac{c'}{c}$. Why?

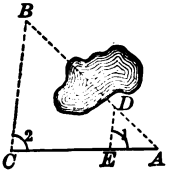
Q. E. F.

327. PROBLEM. *To divide a given segment into a given number of equal parts.*

Construction. The construction is an immediate application of § 326, when $a = b = c$. Q. E. F.

EXERCISE

B is a point visible from A but inaccessible. Required to compute the distance from A to B.

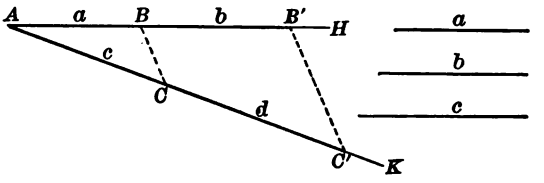


Suggestion. Select some accessible point C from which A and B are both visible. Through E, a point near A and on the line of sight from A to C, draw $ED \parallel CB$ and meeting the line of sight from A to B at D.

Now AE, EC, and AD can be *measured*. Then DB and hence AB can be *computed* by § 323. Show how.

CONSTRUCTION OF A FOURTH PROPORTIONAL

328. PROBLEM. *To construct a fourth proportional to three given line-segments.*



Given three segments a, b, c.

To construct segment d, *such that* $\frac{a}{b}=\frac{c}{d}$.

Construction. On a line AH lay off segments AB and BB' equal to a and b respectively.

Through A draw a line AK, and on it lay off a segment $AC = c$.

Draw BC and through B' draw $B'C' \parallel BC$.

Then CC' is the required segment d, such that

$$\frac{a}{b}=\frac{c}{d}.$$ Why?

Q. E. F.

329. PROBLEM. *To construct a third proportional to two given segments.*

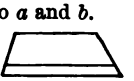
Construction. The construction is an immediate application of § 328, when $b = c$. Q. E. F.

SIGHT WORK

1. If $a = 6$, $b = 10$, $c = 18$, *construct* the fourth proportional to a, b, and c.

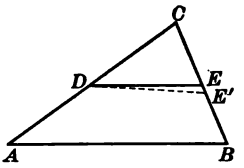
2. If $a = 6$ and $b = 10$, construct the third proportional to a and b.

3. A line parallel to the bases of a trapezoid divides its non-parallel sides proportionally. Prove.



PROPORTIONAL DIVISION OF SIDES OF A TRIANGLE

330. THEOREM IV. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given $\triangle ABC$ **with points** D **and** E **on** AC **and** BC **such that**

$$\frac{CD}{DA}=\frac{CE}{EB}.$$

To prove that $DE \parallel AB$.

Proof: Draw $DE' \parallel AB$.

We need to prove that E' coincides with E.

Now $$\frac{CA}{DA}=\frac{CB}{EB}.$$ § 312 (4)

$$\left(If\ \frac{a}{b}=\frac{c}{d}\ then\ \frac{a+b}{b}=\frac{c+d}{d}.\right)$$

Also $$\frac{CA}{DA}=\frac{CB}{E'B}.$$ § 323

(*If a line is* $\parallel$ *to one side of a* $\triangle$ *and cuts the other two sides, then any two pairs of corresponding segments form a proportion.*)

$\therefore EB = E'B$ and E' coincides with E. § 317

(*If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, then the fourth terms are equal.*)

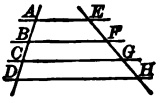
Q. E. D.

331. COROLLARY. *If a line cuts two sides of a triangle so that any two pairs of corresponding segments form a proportion, then the line is parallel to the third side.*

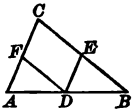
Apply § 324 and then use the theorem.

EXERCISES

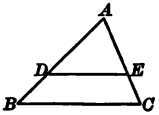
1. In the figure, the horizontal lines are parallel. If $AB = 8$ in., $BC = 7$ in., $CD = 6\frac{1}{2}$ in., and $EF = 10$ in., find the lengths of FG and GH.



2. In the triangle ABC, lines DE and DF are drawn so that $DE \parallel AC$ and $DF \parallel BC$. Prove that $AF : FC = CE : EB$.



3. If in the figure DE is parallel to BC, compute the segments indicated by the blanks in the accompanying table.



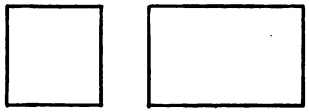
AD	AB	DB	AE	AC	EC
8	12		6		
6	9				7
10		8		18	
240			200	380	
120				100	50
	40	15		30	
	90		40	70	
	800			360	300
		30	20		50

EXPERIMENTAL GEOMETRY

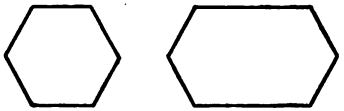
Construct two triangles ABC and $A'B'C'$ so that $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$. Measure the sides with care and compare the three ratios $\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, and $\frac{CA}{C'A'}$. This experiment should be repeated with a number of different pairs of triangles of different shapes and sizes.

Are the triangles in each pair similar? See § 334.

332. **Mutually Equiangular Polygons.** Two polygons in which the angles of one are equal respectively to the angles of the other, taken in order, are said to be *mutually equiangular.*



MUTUALLY EQUIANGULAR.



MUTUALLY EQUIANGULAR.

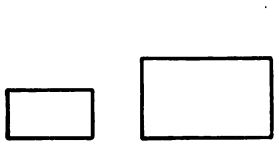
333. **Corresponding Parts of Polygons.** Equal angles of mutually equiangular polygons are called *corresponding angles.*

Sides included between corresponding angles are called *corresponding sides.*

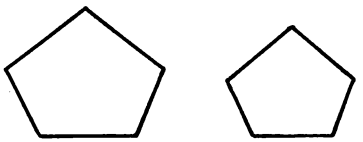
334. **Similar Polygons.** Two polygons are called *similar* if

(1) they are mutually *equiangular*, and if

(2) their corresponding sides are *proportional.*



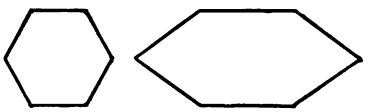
SIMILAR.



SIMILAR.

Two polygons may have property (1) but not (2). See figures at top of page. Or they may have property (2) but not (1). See figures below.

SIDES PROPORTIONAL.



SIDES PROPORTIONAL.

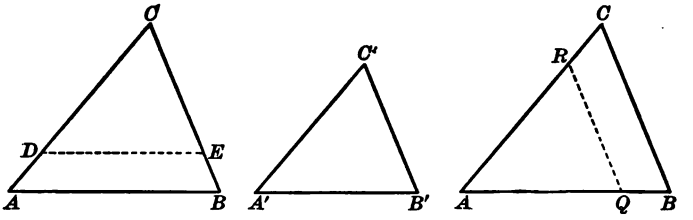
Hence any proof that two polygons are similar must show that both (1) and (2) hold concerning them.

In the case of triangles it will be proved that either property (1) or (2) is sufficient to make them similar.

335. **Ratio of Similitude.** The ratio of two corresponding sides of similar polygons is called their *ratio of similitude.*

EQUIANGULAR TRIANGLES ARE SIMILAR

336. THEOREM V. *Two triangles are similar if they are mutually equiangular.*



Given $\triangle ABC$ and $A'B'C'$ in which $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

To prove that $$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}.$$

Proof: Place $\triangle A'B'C'$ on $\triangle ABC$ with $\angle C'$ upon its equal $\angle C$, making $C'B'$ fall along CB and $C'A'$ fall along CA. Then $A'B'$ will take the position DE.

$\triangle CDE = \triangle C'A'B'$ (Why?) $\therefore \angle CDE = \angle A' = \angle A$.

$\therefore DE \parallel AB$. § 148 (1)

$\therefore \frac{CD}{CA} = \frac{CE}{CB}$. Why?

$\therefore \frac{C'A'}{CA} = \frac{C'B'}{CB}$. Why?

In like manner, placing $\angle A'$ upon its equal $\angle A$, show that

$$\frac{C'A'}{CA} = \frac{A'B'}{AB}.$$

$\therefore \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$. Why?

$\therefore$ in $\triangle ABC$ and $A'B'C'$ the pairs of corresponding sides are proportional.

Hence the triangles are similar. § 334

Q. E. D.

337. COROLLARY 1. *Two triangles are similar if two angles of one are equal respectively to two angles of the other.*

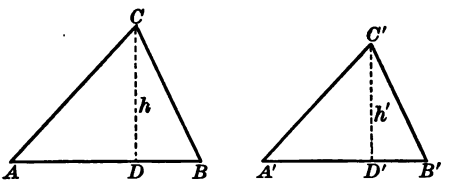
338. COROLLARY 2. *Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.*

339. COROLLARY 3. *If each of two triangles is similar to a third triangle, they are similar to each other.*

340. COROLLARY 4. *If two sides of a triangle are cut by a line parallel to the third side, a triangle is formed which is similar to the given triangle.*

CORRESPONDING ALTITUDES OF SIMILAR TRIANGLES

341. THEOREM VI. *In two similar triangles corresponding altitudes are proportional to any two corresponding sides.*



Given the similar △ *ABC* and *A'B'C'* with *h* and *h'* corresponding altitudes.

To prove that $\frac{h}{h'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{AB}{A'B'}$.

Proof: Since $\triangle ABC \sim \triangle A'B'C'$, $\angle A = \angle A'$. § 334, (1)

$\therefore \triangle ADC \sim \triangle A'D'C'$. § 338

$\therefore \frac{h}{h'} = \frac{AC}{A'C'}$. Why?

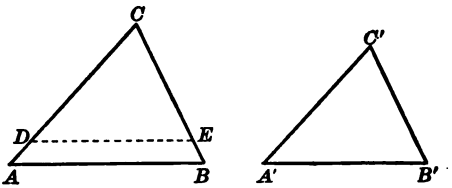
Hence, $\frac{h}{h'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{AB}{A'B'}$. § 334, (2)

(*In similar polygons corresponding sides are proportional.*)

Q. E. D.

CONDITIONS FOR SIMILARITY OF TRIANGLES

342. THEOREM VII. *Two triangles are similar if an angle of one is equal to an angle of the other and the pairs of adjacent sides are proportional.*



Given $\triangle ABC$ **and** $A'B'C'$ **in which** $\angle C = \angle C'$ **and** $\frac{C'A'}{CA} = \frac{C'B'}{CB}$.

To prove that $\triangle ABC$ *and* $A'B'C'$ *are similar.*

Proof: Place $\triangle A'B'C'$ on $\triangle ABC$ with $\angle C'$ on its equal $\angle C$, making $C'B'$ fall along CB and $C'A'$ along CA. Then $A'B'$ will take the position DE.

Then $\frac{CD}{CA} = \frac{CE}{CB}$. Why?

$\therefore DE \parallel AB$. § 331

(*If a line cuts two sides of a* $\triangle$ *so that any two pairs etc.*)

$\therefore \angle A = \angle CDE$ and $\angle B = \angle CED$. Why?

$\therefore \triangle ABC$ and DEC are similar. § 336

(*Two* $\triangle$ *are similar if they are mutually equiangular.*)

$\therefore \triangle ABC$ and $A'B'C'$ are similar. § 339

(*If each of two* $\triangle$ *is similar to a third* $\triangle$, *they are similar to each other.*)

Q. E. D.

SIGHT WORK

1. Give a definition of similar polygons.

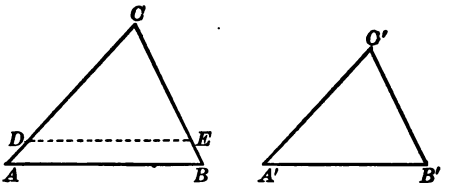
2. Are polygons necessarily similar if they are mutually equiangular? Are triangles?

3. Are polygons necessarily similar if their corresponding sides are proportional? Are triangles? See next page.

4. Are equal triangles similar? Why?

CONDITIONS FOR SIMILARITY OF TRIANGLES

343. THEOREM VIII. *Two triangles are similar if their corresponding sides are proportional.*



Given $\triangle ABC$ **and** $A'B'C'$, **in which** $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$.

To prove that $\triangle ABC$ *and* $A'B'C'$ *are similar.*

Proof: On CA and CB respectively lay off $CD = C'A'$ and $CE = C'B'$, and draw DE.

Then $\frac{CA}{CD} = \frac{CB}{CE}$ since it is given that $\frac{CA}{C'A'} = \frac{CB}{C'B'}$.

Hence $\triangle DEC$ is similar to $\triangle ABC$. § 342

$$\therefore \frac{CA}{AB} = \frac{CD}{DE}. \qquad \text{§ 334}$$

But $CD = C'A'$, $\therefore \frac{CA}{AB} = \frac{C'A'}{DE}$. (1) Ax. 14, § 71

Now it is given that $\frac{CA}{C'A'} = \frac{AB}{A'B'}$, Hyp.

from which $\frac{CA}{AB} = \frac{C'A'}{A'B'}$. (2) Why?

$\therefore$ from (1) and (2), $DE = A'B'$. § 317

$$\therefore \triangle DEC = \triangle A'B'C'.$$

(*Since* $CD = C'A'$, $CE = C'B'$, *and* $DE = A'B'$.)

$\therefore \triangle ABC$ is similar to $\triangle A'B'C'$.

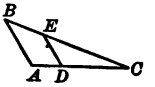
(*Since* $\triangle A'B'C' = \triangle DEC$ *and* $\triangle DEC$ *is similar to* $\triangle ABC$.)

Q. E. D.

EXERCISES

1. Show by similar triangles that the segment joining the mid-points of two sides of a triangle is equal to one half the third side.

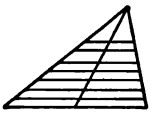
2. In the figure, $DE \parallel AB$ and $\frac{CD}{DA} = \frac{5}{2}$. If $DE = 4$ in., find AB.



3. In the above figure, if $AC = 14$ in., $CD = 10$ in., $BE = 6$ in., $CE = 15$ in., is $DE \parallel AB$? Why?

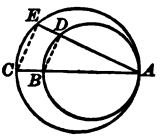
4. In similar triangles the ratio of the bisectors of corresponding angles equals the ratio of similitude of the triangles.

5. A median upon one side of a triangle bisects all segments parallel to that side and terminated by the other two sides.



6. In similar triangles the ratio of the medians upon corresponding sides equals the ratio of similitude of the triangles.

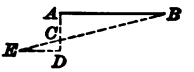
7. If two circles are tangent internally then all chords of the larger circle through the point of tangency are divided in the same ratio by the smaller circle.



Suggestion. Draw the diameter AC and connect points as shown in the figure. Prove $DB \parallel EC$. Compare Example 2, page 160, for the proof that $DB \parallel EC$.

8. To measure indirectly the distance from an accessible point A to an inaccessible point B.

Suggestion. Construct $AD \perp$ the line of sight from A to B, and $ED \perp AD$. Let C be the point on AD which lies in line with E and B.



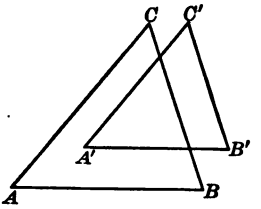
Now show that $\triangle EDC$ and BAC are mutually equiangular and hence similar. What segments need to be measured in order to compute AB?

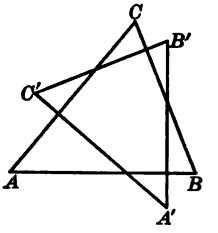
9. Through a point P within an angle to construct a segment terminated by its sides which shall be divided in a given ratio by the point P.

Suggestion. From the vertex A lay off AB and BC so that $AB : BC =$ the given ratio. Then the segment through C and P is the one required.

TRIANGLES WITH SIDES PERPENDICULAR OR PARALLEL

344. THEOREM IX. *Two triangles are similar if the sides of one are respectively parallel, or respectively perpendicular, to the sides of the other.*





(1) **Given** $\triangle ABC$ **and** $A'B'C'$ **in which** $AB \parallel A'B'$, $BC \parallel B'C'$, **and** $CA \parallel C'A'$.

To prove that $\triangle ABC$ *and* $A'B'C'$ *are similar.*

Proof: $\angle A$ and A' are either equal or supplementary. Likewise $\angle B$ and B', C and C'. § 161

(*Two angles whose sides are* ∥ *each to each are equal or supplementary.*)

Suppose that two pairs of these angles are supplementary, as $\angle A$, $\angle A'$ and $\angle B$, $\angle B$ and only one pair equal.

Then $\angle A + \angle A' + \angle B + \angle B' = 4$ rt. $\angle$s. Why?

But this is impossible, because the sum of all the $\angle$s of $\triangle ABC$ and $A'B'C'$ is only 4 rt. $\angle$s.

$\therefore$ at least two pairs of these angles must be equal.

$\therefore$ $\triangle ABC$ is similar to $\triangle A'B'C'$. § 337

(*Two* $\triangle$ *are similar if two* $\angle$ *in one are equal, etc.*)

(2) **Given** $\triangle ABC$ **and** $A'B'C'$ **in which** $AB \perp A'B'$, $BC \perp B'C'$, $CA \perp C'A'$.

To prove that $\triangle ABC$ *and* $A'B'C'$ *are similar.*

Proof: Using § 162 instead of § 161, the proof is exactly like that in (1) above.

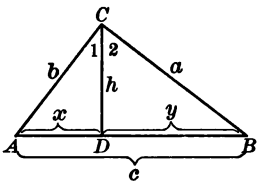
Q. E. D.

ALTITUDE ON THE HYPOTENUSE OF A RIGHT TRIANGLE

345. THEOREM X. *If in a right triangle the altitude on the hypotenuse is drawn, then*

(1) *the altitude is a mean proportional between the segments into which it divides the hypotenuse; and*

(2) *each side of the triangle is a mean proportional between the whole hypotenuse and the adjacent segment.*



Given rt. △ *ABC* with altitude *h*, sides *a*, *b*, *c*, and *x*, *y* segments of the hypotenuse.

To prove that (1) $\frac{x}{h}=\frac{h}{y}$; (2) $\frac{x}{b}=\frac{b}{c}$ *and* $\frac{y}{a}=\frac{a}{c}$.

Proof: (1) △ *ADC* ~ △ *ABC* and △ *BDC* ~ △ *ABC* § 338

(*Two rt.* ▲ *are similar if an acute* ∠ *of one etc.*)

∴ △ *ADC* is similar to △ *BDC*. § 339

(*If each of two* ▲ *is similar to a third* △, *etc.*)

∴ $x : h :: h : y$. § 334

(*Corresponding sides of similar* ▲ *are proportional.*)

(2) Since △ *ABC* is similar to △ *ADC*,

∴ $x : b :: b : c$. Why?

Also since △ *ABC* is similar to △ *BDC*,

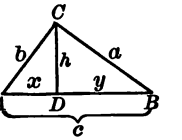
∴ $y : a :: a : c$. Why?

Q. E. D.

346. COROLLARY 1. *If in a right triangle the altitude on the hypotenuse is drawn, then*

(*a*) *the square of the altitude equals the product of the segments of the hypotenuse; and*

(*b*) *the square of either side equals the product of the whole hypotenuse and the adjacent segment.*



From $\frac{x}{h} = \frac{h}{y}$, we have $h^2 = xy$, and from $\frac{x}{b} = \frac{b}{c}$ we have $b^2 = cx$. Likewise $a^2 = cy$.

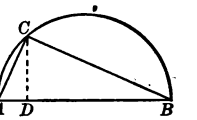
347. COROLLARY 2. *If in a right triangle the altitude on the hypotenuse is drawn, then the squares of the two sides are in the same ratio as the adjacent segments of the hypotenuse.*

Since $a^2 = cy$ and $b^2 = cx$, we have $\frac{a^2}{b^2} = \frac{y}{x}$.

348. COROLLARY 3. *If from a point on a circle a perpendicular is drawn to a diameter, then*

(*a*) *the perpendicular is a mean proportional between the segments of the diameter; and*

(*b*) *each chord is a mean proportional between the whole diameter and the adjacent segment.*



Suggestion. $\angle ACB$ is a right angle because it is inscribed in a semicircle. § 272

SIGHT WORK

1. In the figure of § 346, if $x = 8$ and $y = 18$, find h.

Suggestion. $h^2 = xy = 8 \times 18 = 144$.

2. If, in the same figure, $a = 3$ and $b = 4$, find the ratio of x to y.

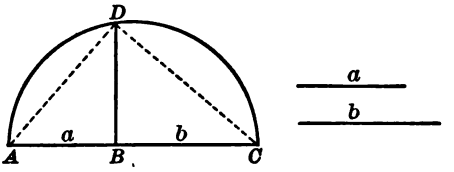
3. If, in the same figure, $c = 5$ and $b = 3$, find x. Also find y if $c = 5$ and $a = 4$.

4. In the same figure find a if $c = 10$ and $y = 8$.

5. If, in the figure of § 348, $AD = 4$ and $CD = 6$, find DB.

CONSTRUCTING A MEAN PROPORTIONAL

349. PROBLEM. *To construct a segment which shall be a mean proportional between two given segments.*



Given segments a and b.

To construct a mean proportional between them.

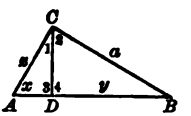
Construction. On a line lay off $AB = a$ and $BC = b$. On AC as a diameter construct a semicircle.

At B draw a perpendicular to AC meeting the circle in D. Then BD is the required mean proportional. § 348

(*If from a point on a ⊙ a ⊥ is drawn to the diameter, etc.*)

Q. E. F.

SIGHT WORK



1. In the figure, $\triangle ABC$ is right angled at C and $CD \perp AB$. Pick out three pairs of similar ▲.

2. In ▲ ACD and ACB of the same figure what are the pairs of equal angles? What are the pairs of corresponding sides?

3. What are the pairs of equal angles and the pairs of corresponding sides in ▲ CDB and ACB? in ▲ ACD and CDB?

4. Read all the proportions derivable from this figure.

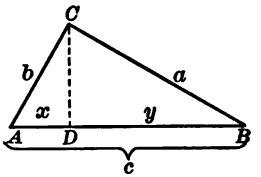
5. In two ▲ ABC and $A'B'C'$, $\angle A = \angle A'$, $AB = 12$, $AC = 14$, $A'B' = 16$, and $A'C' = 18\frac{2}{3}$. Are the ▲ similar? Why?

6. Show that two isosceles triangles are similar if the vertical angle of one is equal to the vertical angle of the other.

7. Are two ▲ ABC and $A'B'C'$ similar, if $AB = 10$, $BC = 14$, $CA = 16$, and $A'B' = 15$, $B'C' = 21$, $C'A' = 24$? Why?

PYTHAGOREAN PROPOSTION

350. THEOREM XI. *In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.*



Given the rt. $\triangle$ *ABC* with sides *a*, *b*, *c*.

To prove that $a^2 + b^2 = c^2$.

Proof: We have $a^2 = cy$ and $b^2 = cx$. § 346

(*If in a rt. $\triangle$ the altitude on the hypotenuse be drawn, then the square etc.*)

Adding, $a^2 + b^2 = cy + cx = c(x + y) = c \cdot c = c^2$,

or $a^2 + b^2 = c^2$. Q. E. D.

351. COROLLARY 1. *The square of one side of a right triangle equals the square of the hypotenuse minus the square of the other side.*

That is, in the above figure, $a^2 = c^2 - b^2$ and $b^2 = c^2 - a^2$.

352. COROLLARY 2. *If a, b, c, and a', b', c' are the sides of two right triangles, c and c' being the hypotenuses, and if $c = c'$ and $a < a'$, then $b > b'$.*

For by § 350, $c^2 = a^2 + b^2$ and $c'^2 = a'^2 + b'^2$.

Also $c^2 = c'^2$ as given.

Then $a^2 + b^2 = a'^2 + b'^2$. Ax. 5, § 71

But it is given that $a^2 < a'^2$.

$\therefore b^2 > b'^2$, Ax. 13, § 71

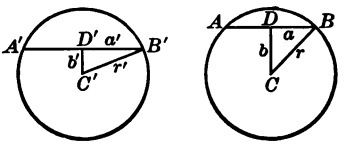
and $\therefore b > b'$.

Q. E. D.

UNEQUAL CHORDS UNEQUALLY DISTANT FROM THE CENTER

353. The proof of the theorems in § 260 follows directly from the preceding corollary.

In the same circle or in equal circles, (1) *if two chords are unequal, the greater chord is nearer the center; and* (2) *conversely, if two chords are unequally distant from the center, the one nearer the center is the greater.*



(1) **Given** $\odot C = \odot C'$ **and chord** $A'B' >$ **chord** AB.

To prove that $A'B'$ *is nearer to* C' *than* AB *is to* C; *that is,* $b' < b$, *or* $b > b'$.

Proof: In the rt. $\triangle CBD$ and $C'B'D'$,

$r = r'$, and $a' > a$, or $a < a'$. (Hyp.)

$\therefore b > b'$. § 352

(2) In the converse we notice that the hypothesis is $r = r'$, and $b' < b$, and we are to prove $a' > a$.

Complete the proof. Q. E. D.

SIGHT WORK

1. Find the square of the hypotenuse of a right triangle if its sides are 3 inches and 4 inches, respectively. Also find the hypotenuse.

2. Find the square of the hypotenuse of a right triangle whose sides are 4 inches and 5 inches. Using the square root sign, express the length of the hypotenuse.

3. One right triangle has sides 5 inches and 8 inches, and another has sides 6 inches and 7 inches. Which has the greater hypotenuse?

EXERCISES

1. The radius of a circle is 5. What is the distance from the center to a chord whose length is 6?

2. In the same circle, what is the length of a chord whose distance from the center is 4?

3. Find the diagonal of a square whose side is 5; whose side is a.

4. What is the side of a square whose diagonal is 8? whose diagonal is d?

5. The hypotenuse of a right isosceles triangle is 12 inches. Find the lengths of its sides.

6. The diagonals of a rhombus are 14 and 10 inches respectively. Find the length of its sides.

Suggestion. Show that the diagonals are perpendicular to each other.

7. What is the radius of a circle if a chord 12 inches long is 9 inches from the center?

EXPERIMENTAL GEOMETRY

1. Construct a large circle and draw chords AB and DE meeting in a point P. Measure PA, PB, PD, and PE with great care. Then find the products $PA \times PB$ and $PD \times PE$. How nearly equal are these products?

Repeat this experiment several times. What statement does this experiment seem to justify?

Note. — Bear in mind that all practical measurements are only approximations. If the circle were absolutely perfect and if the measurements were exact, the products obtained in this experiment would be exactly equal. See § 354.

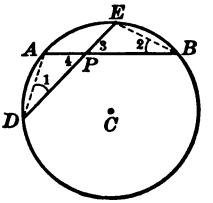
2. From a point P outside a circle draw two secants meeting the circle in A, B and in D, E respectively. Measure PA, PB, PD, and PE. Then find the products $PA \times PB$ and $PE \times PD$. By how much do these products differ?

3. From a point P outside a circle draw a tangent PD and a secant meeting the circle in A and B. Measure and compare the products $PA \times PB$ and $PD \times PD = \overline{PD}^2$. By how much do they differ?

4. What statements seem to be justified by Experiments 2 and 3?

PRODUCT OF SEGMENTS OF A CHORD

354. THEOREM XII. *If two chords of a circle intersect, the product of the segments of one chord is equal to the product of the segments of the other.*



Given ⊙ C **with chords** AB **and** DE **meeting in** P.

To prove that $PA \times PB = PD \times PE$.

Proof: Draw the chords AD and BE.

In ▲ DPA and BPE

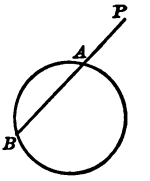
$\angle 4 = \angle 3$ and $\angle 1 = \angle 2$.	Why?
$\therefore \triangle DPA$ is similar to $\triangle BPE$.	§ 337
$\therefore \frac{PA}{PD} = \frac{PE}{PB}$.	Why?
$\therefore PA \times PB = PD \times PE$.	§ 312, (1)

Note. — This proves the *exact* equality which was *approximated* in the experiment on page 191. Q. E. D.

355. COROLLARY 1. *If in a fixed circle a chord AB is made to swing about the fixed interior point P as a pivot, the product $AP \times PB$ remains fixed.*

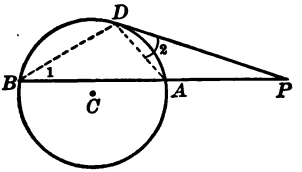
356. COROLLARY 2. *If two chords meet within a circle, their segments form a proportion, in which those of one chord are the means and those of the other the extremes.*

357. Secant and External Segment. If a secant of a circle is drawn from a point P outside it, meeting the circle in the points A and B, then PB is called the *whole secant* and PA the *external* segment, provided A lies between B and P.



THE SQUARE OF A TANGENT

358. THEOREM XIII. *If a secant and a tangent to a circle meet in a point, then the square of the tangent from this point is equal to the product of the whole secant and its external segment.*



Given ⊙ C **with secant** PAB **and tangent** PD.

To prove that $PB \times PA = \overline{PD}^2$.

Proof: Draw BD and AD.

In ▲ PAD and PBD, $\angle P = \angle P$, $\angle 1 = \angle 2$. §§ 270, 275

$\therefore \triangle PAD$ is similar to $\triangle PBD$, Why?

$$\therefore \frac{PB}{PD} = \frac{PD}{PA},$$

and $$PB \times PA = \overline{PD}^2.$$ Why?

Q. E. D.

359. COROLLARY 1. *If from a point outside a circle two secants are drawn, the product of one whole secant and its external segment equals the product of the other secant and its external segment.*

Proof: The product of each secant and its external segment is equal to the square of the tangent from their point of intersection.

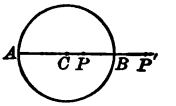
360. COROLLARY 2. *If in a fixed circle a secant PAB is made to revolve about a fixed exterior point P as a pivot, then the product $PB \times PA$ remains fixed.*

SIGHT WORK

1. If the diameter AB of a circle is prolonged and a point P is taken on this line, the product of the distances PA and PB is sometimes called, in higher mathematics, *the power of the point P with respect to the circle.*

What is $PA \times PB$, if the point P is at A or B?

What is $PA \times PB$, if P is at C?



How does the value of $PA \times PB$ change as P moves from the center toward the circle?

How does $PA \times PB$ change as P moves away *outside* the circle?

2. What theorems are included in the following statement: "*From a point P in a plane a line is drawn cutting a circle in A and B. Then the product $PA \times PB$ is the same for all such lines through P*"?

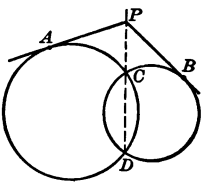
3. If, in the figure of § 358, a square is constructed on PD as a side, and a rectangle with PB as base and PA as altitude, compare their areas as the secant revolves about P as a pivot.

4. A chord 12 in. long is 8 in. from the center. Find the diameter.

5. A chord AB is 6 in. long. It is produced through B to P so that PB is 18 in. Find the length of the tangent to the circle from P.

EXERCISES

1. The lengths of the tangents to two intersecting circles from a point on the common chord extended are equal.

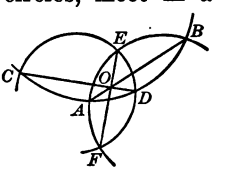


Suggestion. $\overline{PA}^2 = PC \times PD$ and $\overline{PB}^2 = PC \times PD$.

2. The common chord (extended) of two intersecting circles bisects the common tangents.

3. If two segments AB and CD intersect in P and if $PA \times PB = PC \times PD$, then a circle may be passed through the four points A, B, C, D.

4. If each of three circles cuts the other two, then the three chords, each of which is common to two circles, meet in a point.



Suggestion. Let AB and CD meet in O. Draw FO, and prolong it to meet $\overset{\frown}{CD}$ in E' and $\overset{\frown}{AB}$ in E''. Now show that E' and E'' must coincide with E, the point common to the two arcs.

EXERCISES IN COMPUTATION

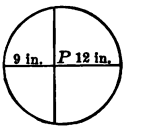
1. A point P is 8 in. from the center of a circle whose radius is 4 in. Any secant is drawn from P, cutting the circle. Find the product of the whole secant and its external segment.

2. From the same point without a circle two secants are drawn. If one whole secant and its external segment are 14 and 5, respectively, and the other external segment is 7, find the other whole secant.

3. A point P is 10 in. from the center of circle whose radius is 6 in. Find the length of the tangent from P.

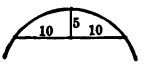
4. A tangent from P to a circle is 7 in., and the external segment of a secant is 4 in. Find the length of the secant.

5. A point P divides the diameter of a circle into two segments of 9 in. and 12 in., respectively. Find the length of the chord perpendicular to the diameter at P.

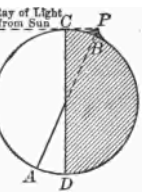


6. A point P on a diameter divides it into segments of 4 in. and 16 in., respectively. What is the length of a chord through P perpendicular to this diameter?

7. In a fragment of a circle a chord 20 in. long is drawn. If the greatest perpendicular distance from a point on the arc to the chord is 5 in., find the diameter of the circle.



8. The height BP of a mountain on the moon is found as follows: the distance PC is measured as the top of the mountain catches the light from the rising sun. The distance AB is known. If AB is 2163 miles and CP is found to be 85 miles, what is the length of BP?

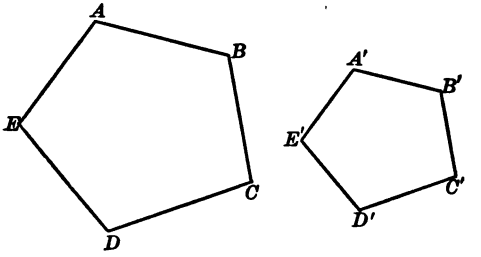


This method was first used by Galileo about 1610.

9. A mountain on the earth can just be seen at a distance of 100 miles at sea. What is its height if the diameter of the earth is 8000 miles?

RATIO OF PERIMETERS OF SIMILAR POLYGONS

361. THEOREM XIV. *The perimeters of two similar polygons are in the same ratio as any two corresponding sides.*



Given the similar polygons $ABCDE$ **and** $A'B'C'D'E'$ **with perimeters** p **and** p'.

To prove that $$\frac{p}{p'} = \frac{AB}{A'B'}.$$

Proof: By definition of similar polygons § 334 (2)

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$$

$$\therefore \frac{AB + BC + CD + DE + EA}{A'B' + B'C' + C'D' + D'E' + E'A'} = \frac{AB}{A'B'}. \quad \text{§ 318}$$

(*In a series of equal ratios the sum of the antecedents is to the sum of, etc.*)

Q. E. D.

That is, $$\frac{p}{p'} = \frac{AB}{A'B'}.$$

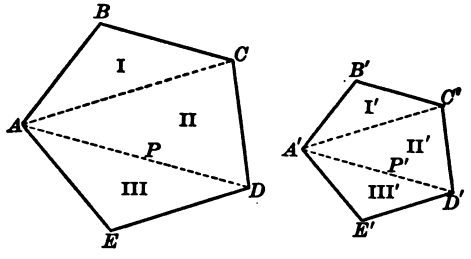
SIGHT WORK

1. If in the above figure $AB = 6$ inches and $A'B' = 4$ inches, and the perimeter of the smaller polygon is 18 inches, find the perimeter of the larger.

2. If the perimeters of the above figures are 47 and 72, respectively, and if $AB = 12$, find $A'B'$.

TRIANGLES FORMING SIMILAR POLYGONS

362. THEOREM XV. *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, then the polygons are similar.*



Given polygons P and P', composed of $\triangle$ I, II, III, and I', II', III', such that $\triangle$ I $\sim$ $\triangle$ I', $\triangle$ II $\sim$ $\triangle$ II', $\triangle$ III $\sim$ $\triangle$ III'.

To prove that $P \sim P'$.

Proof: $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CA}{C'A'}$. ($\triangle$ I $\sim$ $\triangle$ I')

$\dfrac{CA}{C'A'} = \dfrac{CD}{C'D'}$. ($\triangle$ II $\sim$ $\triangle$ II')

$\therefore \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$. Ax. 5, § 71

Continuing thus, we prove

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}. \quad (1)$$

Also $\angle BCA = \angle B'C'A'$, $\angle ACD = \angle A'C'D'$. Why?

$\therefore \angle BCA + \angle ACD = \angle B'C'A' + \angle A'C'D'$, Why?

or $\angle C = \angle C'$.

Continuing thus, we show that

$\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$, $\angle E = \angle E'$. (2)

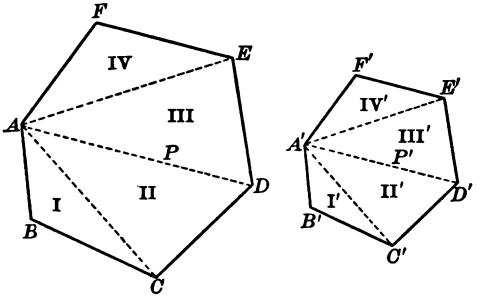
From (1) and (2)

$ABCDE \sim A'B'C'D'E'$. § 334

Q. E. D.

THE DECOMPOSITION OF SIMILAR POLYGONS

363. THEOREM XVI. *Two similar polygons can be divided into the same number of triangles, similar each to each and similarly placed.*



Given similar polygons P and P' with diagonals drawn from the vertices A and A'.

To prove that $\triangle$ I $\sim \triangle$ I', $\triangle$ II $\sim \triangle$ II', $\triangle$ III $\sim \triangle$ III', $\triangle$ IV $\sim \triangle$ IV'.

Proof: $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$ and $\angle B = \angle B'$. $(P \sim P')$

$\therefore \triangle$ I is similar to $\triangle$ I'. § 342

(*Two* $\triangle$ *are similar if an* $\angle$ *of one is equal to an* $\angle$ *of the other, etc.*)

Then $\dfrac{BC}{B'C'} = \dfrac{CA}{C'A'}$, $(\triangle \text{ I} \sim \triangle \text{ I}')$

and $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$. $(P \sim P')$

$\therefore \dfrac{CA}{C'A'} = \dfrac{CD}{C'D'}$. Why?

Also $\angle BCD = \angle B'C'D'$ and $\angle BCA = \angle B'C'A'$. Why?

$\therefore \angle ACD = A'C'D'$. Ax. 2, § 71

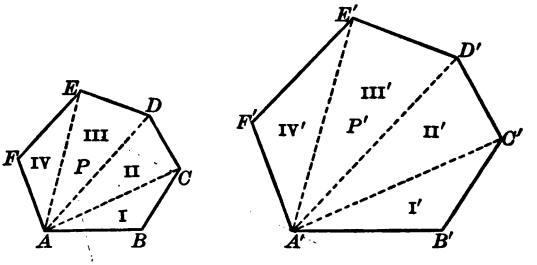
$\therefore \triangle$ II is similar to $\triangle$ II'. Why?

In like manner prove $\triangle$ III $\sim \triangle$ III' and $\triangle$ IV $\sim \triangle$ IV'.

Q. E. D.

CONSTRUCTION OF SIMILAR POLYGONS

364. PROBLEM. *Upon a given segment as a side to construct a polygon similar to a given polygon.*



Given polygon P and segment $A'B'$.

To construct polygon $P' \sim P$ on $A'B'$ corresponding to AB.

Construction. At B' construct $\angle B' = \angle B$.

Lay off $B'C'$ so that $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$. § 328

Then $\triangle$ I$' \sim \triangle$ I. Why?

At C' construct $\angle A'C'D' = \angle ACD$,

and lay off $C'D'$ so that

$$\frac{AC}{A'C'} = \frac{CD}{C'D'}.$$

Then $\triangle$ II$' \sim \triangle$ II. Why?

Proceeding in this manner we construct $\triangle$ III$'$ and IV$'$ so that $\triangle$ III$' \sim \triangle$ III and $\triangle$ IV$' \sim \triangle$ IV.

$\therefore A'B'C'D'E'F' \sim ABCDEF.$ § 362

(*If two polygons are composed of the same number of* $\triangle$, *similar each to each, etc.*)

SIGHT WORK

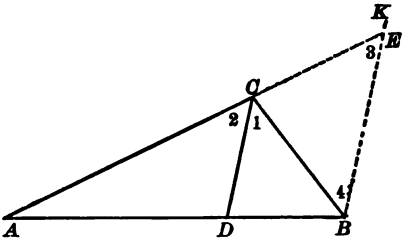
1. How are theorems XV and XVI related?

2. Is it possible to construct a polygon similar to a given polygon if the ratio of their perimeters is given? How? See § 361.

SUPPLEMENTARY THEOREMS AND PROBLEMS

INTERNAL DIVISION OF A SIDE OF A TRIANGLE *

365. THEOREM. *The bisector of an angle of a triangle divides the opposite side into segments whose ratio is the same as that of the adjacent sides.*



Given $\triangle ABC$ **with** CD **bisecting** $\angle C$.

To prove that $$\frac{AC}{BC}=\frac{AD}{BD}.$$

Proof: Draw $BK \parallel DC$. Prolong AC to meet BK at E.

Then $\angle 2=\angle 1=\angle 4$ and $\angle 2=\angle 3$. Why?

$\therefore \angle 4=\angle 3$ and $CE=CB$. Why?

But $\frac{AC}{CE}=\frac{AD}{DB}$. $\therefore \frac{AC}{BC}=\frac{AD}{BD}$. Why? Q. E. D.

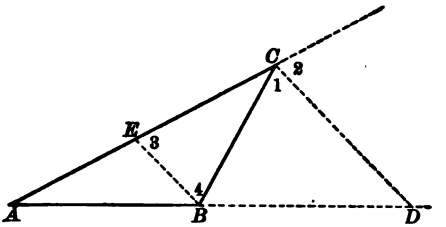
366. Internal and External Division. A line-segment AB is said to be divided *internally* in a given ratio $\frac{m}{n}$ by a point C lying on the segment, if $\frac{AC}{BC}=\frac{m}{n}$.

A line-segment is said to be divided *externally* in a given ratio $\frac{r}{s}$ by a point C' lying on AB produced, if $\frac{AC'}{BC'}=\frac{r}{s}$.

* Pages 200–219 inclusive contain supplementary matter, some or all of which may be omitted without destroying the continuity.

EXTERNAL DIVISION OF A SIDE OF A TRIANGLE

367. THEOREM. *A line which bisects an exterior angle of a triangle divides the opposite side externally into two segments whose ratio is the same as that of the adjacent sides of the triangle.*



Given $\triangle ABC$ **with** CD **the bisector of the exterior angle at** C.

To prove that $$\frac{AD}{BD}=\frac{AC}{BC}.$$

Proof: Draw $BE \parallel DC$. Then $\frac{AD}{BD}=\frac{AC}{EC}$. Why?

Now show $\angle 3 = \angle 2$, $\angle 2 = \angle 1 = \angle 4$, and $EC = BC$.

Complete the proof.

368. **Harmonic Division.** A line-segment AB is said to be divided *harmonically* if the points D and D', lying respectively on AB and on AB produced, are such that $\frac{AD}{BD}=\frac{AD'}{BD'}$.

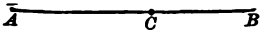
369. PROBLEM. *To divide a given segment harmonically.*

Suggestion. On the given segment AB as a base construct a triangle ABC not isosceles. Let CD and CD' be the bisectors of the internal and external angles at C.

Prove $$\frac{AD}{BD}=\frac{AC}{BC}=\frac{AD'}{BD'}.$$

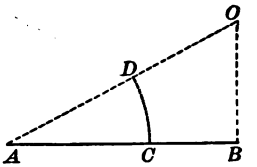
DIVISION IN EXTREME AND MEAN RATIO

370. **Extreme and Mean Ratio.** A segment is said to be divided in *extreme and mean ratio* by a point on it, provided the ratio of the whole segment to the larger part equals the ratio of the larger part to the smaller.



E.g. the segment AB is divided in extreme and mean ratio by the point C if $\frac{AB}{AC}=\frac{AC}{CB}$.

371. PROBLEM. *To divide a given line-segment in extreme and mean ratio.*



Solution. At one end B erect $BO \perp AB$ and make $BO=\frac{1}{2}AB$. Draw OA. On OA take $OD=OB$ and on AB take $AC=AD$. Then C is the required point of division.

Proof: Let a be the length of AB. Then $OB=\frac{a}{2}$.

Hence $\overline{AO}^2=\overline{AB}^2+\overline{OB}^2=a^2+\frac{a^2}{4}=\frac{5}{4}a^2$, or $AO=\frac{a}{2}\sqrt{5}$.

Then $AC=AD=AO-DO=\frac{a}{2}\sqrt{5}-\frac{a}{2}=\frac{a}{2}\left(\sqrt{5}-1\right)$,

and $BC=AB-AC=a-\frac{a}{2}\left(\sqrt{5}-1\right)=\frac{a}{2}\left(3-\sqrt{5}\right)$.

Substituting these values in $\frac{AB}{AC}$ and $\frac{AC}{CB}$, and simplifying,

we have $\frac{2}{\sqrt{5}-1}$ and $\frac{\sqrt{5}-1}{3-\sqrt{5}}$.

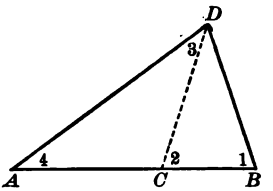
Prove these equal by rationalizing denominators.

Hence $\frac{AB}{AC}=\frac{AC}{CB}$. Ax. 5, § 71

Q. E. F.

CONSTRUCTION OF THE ANGLES 36° AND 72°

372. PROBLEM. *To construct with ruler and compasses angles of* 36° *and* 72°.



Construction. On a given segment AB, determine a point C such that $\frac{AB}{AC}=\frac{AC}{BC}$. § 371

Construct $\triangle ABD$ so that $AD=AB$, $BD=AC$.

Then $\angle 1=72°$ and $\angle 4=36°$.

Proof: Draw DC. Then in $\triangle ABD$ and CDB, $\angle 1$ is common and $\frac{AB}{BD}=\frac{BD}{BC}$, since $BD=AC$, and $\frac{AB}{AC}=\frac{AC}{BC}$ as given.

$\therefore \triangle ABD$ is similar to $\triangle CDB$. Why?

Hence $\triangle CBD$ is isosceles, since $\triangle ABD$ was made so.

Then $BD=DC=AC$

and $\angle 1=\angle 2$, $\angle 3=\angle 4$.

Also $\angle 2=\angle 3+\angle 4=2\angle 4=\angle 1$. § 160

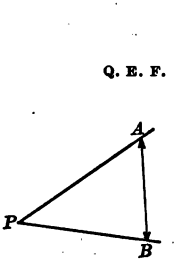
Hence in $\triangle ABD$, $\angle 1=2\angle 4$ and $\angle ADB=2\angle 4$.

Then $2\angle 4+2\angle 4+\angle 4=2$ rt. $\measuredangle$.

That is, $\angle 4=\frac{1}{5}$ of 2 rt. $\measuredangle=36°$,

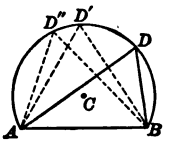
and $\angle 1=2\angle 4=72°$. Q. E. F.

373. **Angle Subtended by an Object.** An object AB is said to *subtend an angle APB* from a point P if the lines PA and PB are supposed to be drawn from P to the extremities of the object.



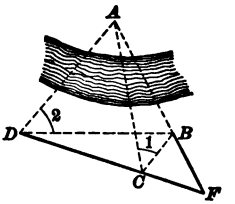
EXERCISES

1. A triangle has a fixed base and a constant vertex angle. Show that the locus of the vertex consists of two arcs of equal circles whose end-points are the extremities of the base. See Examples 8 and 9, page 145, and also § 287.



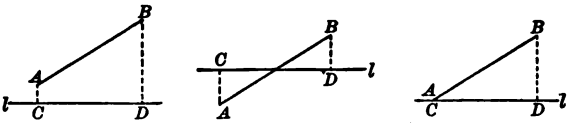
It follows that if a segment AB subtends the same angles from D and from D', D and D' being on the same side of the segment AB, then a circle may be passed through A, B, D, D'.

2. Not having any instruments, an engineer proceeds as follows to obtain approximately the distance from an accessible point B to an inaccessible point A. Walking from B along the line AB he takes 50 steps to F. Then he walks in a convenient direction 50 steps to C, and notes that AB subtends a certain angle, $\angle 1$. He then proceeds along the same straight line until he reaches a point D at which AB again subtends the same angle as at C. That is, $\angle 1 = \angle 2$. Does he conclude correctly that $DC = AB$?



Suggestion. Use Example 1 and then § 359. $\angle 1$ and $\angle 2$ can be measured roughly by sighting across a pencil held at arm's length.

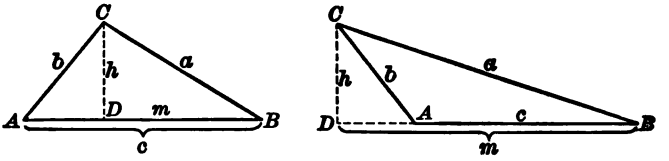
374. Projection of a Segment on a Line. A segment AB is said to be projected on a line l if perpendiculars from A and B are drawn to l. If these perpendiculars meet l in C and D, then CD is called the projection of AB on l.



In case one end-point of segment AB, as A, is on l, then the points A and C coincide.

SQUARE OF SIDE OPPOSITE AN ACUTE ANGLE

375. THEOREM. *The square of the side opposite an acute angle of a triangle is equal to the sum of the squares of the other two sides minus twice the product of one of these sides and the projection of the other upon it.*



Outline of proof. In either figure let $\angle B$ be the given acute angle, and in each case BD is the projection of BC upon AB. Call this projection m.

We are to prove that $b^2 = a^2 + c^2 - 2cm$.

In the left figure, $b^2 = h^2 + (c - m)^2$. (1)

In the right figure, $b^2 = h^2 + (m - c)^2$. (2)

In either case $h^2 = a^2 - m^2$. (3)

Substituting in (1), $b^2 = a^2 - m^2 + c^2 - 2cm + m^2$

$= a^2 + c^2 - 2cm$.

Substituting in (2), $b^2 = a^2 - m^2 + m^2 - 2cm + c^2$

$= a^2 + c^2 - 2cm$. Q. E. D.

SIGHT WORK

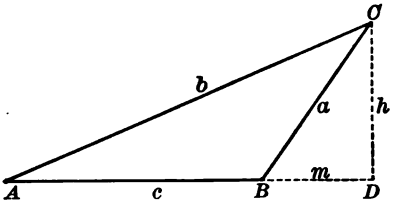
1. If a segment AB is perpendicular to a line l, what is the length of the projection of AB on l? What if $AB \parallel l$?

2. If in the above figure $\angle B$ were a right angle, what would the equation $b^2 = a^2 + c^2 - 2cm$ reduce to? Explain this and compare the result with the theorem of § 350.

3. If in the right-hand figure above $a = 10$, $c = 8$, and $m = 9$, find b.

SQUARE OF SIDE OPPOSITE AN OBTUSE ANGLE

376. THEOREM. *The square of the side opposite an obtuse angle of a triangle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides and the projection of the other upon it.*



Outline of proof. Let $\angle B$ be the given obtuse angle and BD the projection of BC upon AB. Call this projection m.

We are to prove that $b^2 = a^2 + c^2 + 2\,cm$.

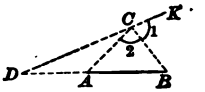
In the figure $b^2 = h^2 + (c + m)^2$ and $h^2 = a^2 - m^2$.

$$\therefore\ b^2 = a^2 - m^2 + c^2 + 2\,cm + m^2 = a^2 + c^2 + 2\,cm.$$

Q. E. D.

EXERCISES

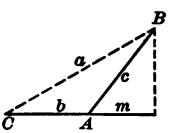
1. To find the distance from an accessible point A to an inaccessible point B by means of § 367.



Suggestion. Through C, a point where A and B are both visible, draw CK making $\angle 1 = \angle 2$. Produce KC to a point D on the line BA extended.

What lines must now be measured in order to compute AB?

2. Find the length of the projection m of a segment AB on a line through A, using the figure shown to the right.



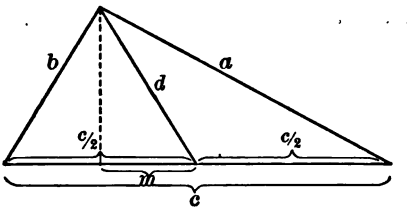
Measure AB, AC, and BC. Put $AB = c$, $AC = b$, $BC = a$. From § 376 $a^2 = b^2 + c^2 + 2mb$, whence

$$m = \frac{a^2 - b^2 - c^2}{2b}.$$

If $BC = 15$, $AC = 8$, and $AB = 9$, find m.

SUM OF SQUARES OF TWO SIDES

377. THEOREM. *The sum of the squares of two sides of any triangle is equal to half the square of the third side plus twice the square of the median drawn to that side.*



Outline of proof. In the figure,

$$b^2 = d^2 + \left(\frac{c}{2}\right)^2 - 2m\left(\frac{c}{2}\right). \quad (1)$$ Why?

$$a^2 = d^2 + \left(\frac{c}{2}\right)^2 + 2m\left(\frac{c}{2}\right). \quad (2)$$ Why?

Adding (1) and (2), $a^2 + b^2 = 2d^2 + \frac{c^2}{2}$.

State the proof fully.

378. COROLLARY. *The difference of the squares of two sides of a triangle equals twice the product of the third side and the projection of the median on it.*

Subtracting equation (1) from (2) in § 377,

$$a^2 - b^2 = 4m\frac{c}{2} = 2mc.$$

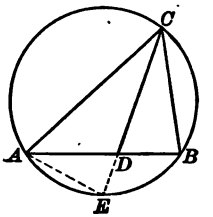
EXERCISE

Find the locus of a point such that the sum of the squares of its distances from two given points is a constant.

Solution. Let the vertex of the triangle in the above figure be the moving point and the extremities of the base the fixed points. Then $a^2 + b^2 = 2d^2 + \frac{c^2}{2} = k$ (a constant). But c is a constant and d is, therefore, a constant. Hence, the locus of the point is a circle with radius d.

THE SQUARE OF THE ANGLE BISECTOR

379. THEOREM. *The square of the bisector of an angle of a triangle is equal to the product of the two adjacent sides minus the product of the segments of the opposite side.*



Outline of proof: Produce the bisector CD of the given angle to meet the circumscribed circle in E.

We are to prove that

$$\overline{CD}^2 = AC \times CB - AD \times DB.$$

Give reasons for each of the following:

$$\triangle BDC \sim \triangle AEC. \tag{1}$$

$$AC \cdot BC = CD \cdot CE. \tag{2}$$

But $$CD \cdot CE = CD(CD + DE) = \overline{CD}^2 + CD \cdot DE, \tag{3}$$

and $$CD \cdot DE = AD \cdot DB. \tag{4}$$

From (3) $$\overline{CD}^2 = CD \cdot CE - CD \cdot DE.$$

Substituting from (2) and (4),

$$\overline{CD}^2 = AC \cdot BC - AD \cdot DB.$$ Q. E. D.

EXERCISES

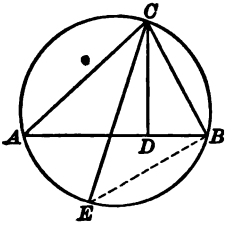
1. Find the length of the bisector of $\angle A$ in the triangle ABC if $AB = 10$, $BC = 12$, and $CA = 14$.

Suggestion. First find the segments into which the bisector of $\angle A$ divides the side BC, using § 365 and § 314. Finally use § 379.

2. Find the lengths of the three bisectors of the angles of a triangle whose sides are 40, 50, and 60, respectively.

PRODUCT OF TWO SIDES OF A TRIANGLE

380. THEOREM. *The product of two sides of a triangle is equal to the product of the altitude from the vertex in which these sides meet and the diameter of the circumscribed circle.*



Outline of proof: Using the figure, show that:

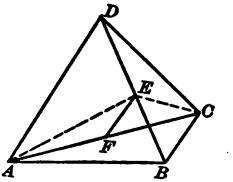
(1) $\triangle ACD$ is similar to $\triangle EBC$.

(2) $AC \cdot BC = CE \cdot CD$. Q. E. D.

EXERCISES

1. If the diagonals of a quadrilateral are perpendicular to each other, the sums of the squares of the pairs of opposite sides are equal.

2. The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square on the line connecting their middle points.



Suggestion. AE, CE, EF are medians of $\triangle ABD$, BCD, and CEA, respectively. Then by § 377

$$\overline{AD}^2 + \overline{AB}^2 = \frac{\overline{BD}^2}{2} + 2\,\overline{AE}^2 \text{ and } \overline{BC}^2 + \overline{CD}^2 = \frac{\overline{BD}^2}{2} + 2\,\overline{CE}^2.$$

Apply the same theorem to $\triangle AEC$ and complete the proof.

3. In any parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of the four sides.

SINES, COSINES, AND TANGENTS

381. Sine. Cosine. Tangent. In a right $\triangle ABC$, right angled at C, the ratio $\frac{BC}{AB}$ is called the **sine** of $\angle A$, the ratio $\frac{AC}{AB}$ is called the **cosine** of $\angle A$ and the ratio $\frac{BC}{AC}$ is called the **tangent** of $\angle A$.

The sine, cosine, and tangent of $\angle A$ are written

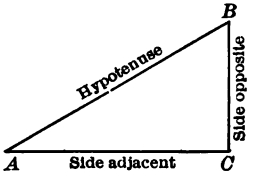
$$\sin A,\ \cos A,\ \tan A.$$

These definitions are also written:

$$\textbf{sin } A = \frac{\textit{side opposite}}{\textit{hypotenuse}}.$$

$$\textbf{cos } A = \frac{\textit{side adjacent}}{\textit{hypotenuse}}.$$

$$\textbf{tan } A = \frac{\textit{side opposite}}{\textit{side adjacent}}.$$

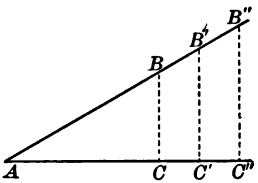


In the adjoining figure, show why

$$\sin A = \frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''};$$

$$\cos B = \frac{AC}{AB} = \frac{AC'}{AB'} = \frac{AC''}{AB''};$$

$$\tan A = \frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{B''C''}{AC''}.$$



Hence, we see that the sine, cosine, and tangent of an angle depend only upon the *size* of the angle and not upon the *length* of its sides.

382. The table on the opposite page gives the sine, cosine, and tangent for angles of whole degrees from 1° to 89°.

If an acute angle is given, the sine, cosine and tangent can be read at sight from this table. Thus, sin 35° = .574, cos 47° = .682, tan 53° = 1.327.

EXERCISES

From the table find sin 37°, cos 44°, tan 73°, sin 81°, cos 67°, tan 14°.

TABLE OF SINES, COSINES, AND TANGENTS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.017	.9998	.017	45°	.707	.707	1.000
2°	.035	.9994	.035	46°	.719	.695	1.036
3°	.052	.9986	.052	47°	.731	.682	1 072
4°	.070	.9976	.070	48°	.743	.669	1.111
5°	.087	.996	.087	49°	.755	.656	1.150
6°	.105	.995	.105	50°	.766	.643	1.192
7°	.122	.993	.123	51°	.777	.629	1.235
8°	.139	.990	.141	52°	.788	.616	1.280
9°	.156	.988	.158	53°	.799	.602	1.327
10°	.174	.985	.176	54°	.809	.588	1.376
11°	.191	.982	.194	55°	.819	.574	1.428
12°	.208	.978	.213	56°	.829	.559	1.483
13°	.225	.974	.231	57°	.839	.545	1.540
14°	.242	.970	.249	58°	.848	.530	1.600
15°	.259	.966	.268	59°	.857	.515	1.664
16°	.276	.961	.287	60°	.866	.500	1.732
17°	.292	.956	.306	61°	.875	.485	1.804
18°	.309	.951	.325	62°	.883	.469	1.881
19°	.326	.946	.344	63°	.891	.454	1.963
20°	.342	.940	.364	64°	.899	.438	2.050
21°	.358	.934	.384	65°	.906	.423	2.145
22°	.375	.927	.404	66°	.914	.407	2.246
23°	.391	.921	.424	67°	.921	.391	2.356
24°	.407	.914	.445	68°	.927	.375	2.475
25°	.423	.906	.466	69°	.934	.358	2.605
26°	.438	.899	.488	70°	.940	.342	2.747
27°	.454	.891	.510	71°	.946	.326	2.904
28°	.469	.883	.532	72°	.951	.309	3.078
29°	.485	.875	.554	73°	.956	.292	3.271
30°	.500	.866	.577	74°	.961	.276	3.487
31°	.515	.857	.601	75°	.966	.259	3.732
32°	.530	.848	.625	76°	.970	.242	4.011
33°	.545	.839	.649	77°	.974	.225	4.331
34°	.559	.829	.675	78°	.978	.208	4.705
35°	.574	.819	.700	79°	.982	.191	5.145
36°	.588	.809	.727	80°	.985	.174	5.671
37°	.602	.799	.754	81°	.988	.156	6.314
38°	.616	.788	.781	82°	.990	.139	7.115
39°	.629	.777	.810	83°	.993	.122	8.144
40°	.643	.766	.839	84°	.995	.105	9.514
41°	.656	.755	.869	85°	.996	.087	11.430
42°	.669	.743	.900	86°	.9976	.070	14.300
43°	.682	.731	.933	87°	.9986	.052	19.081
44°	.695	.719	.966	88°	.9994	.035	28.636
45°	.707	.707	1.000	89°	.9998	.017	57.290

FINDING HEIGHTS AND DISTANCES

383. Formulas used in Finding Parts of Right Triangles.

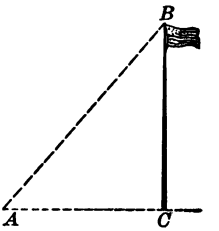
From $\sin A = \frac{BC}{AB}$ we have $BC = AB \sin A$, and $AB = \frac{BC}{\sin A}$.

From $\cos A = \frac{AC}{AB}$ we have $AC = AB \cos A$, and $AB = \frac{AC}{\cos A}$.

From $\tan A = \frac{BC}{AC}$ we have $BC = AC \tan A$, and $AC = \frac{BC}{\tan A}$.

These formulas may be used to solve a number of important problems.

Example 1. Find the height of the flagpole shown in the figure.



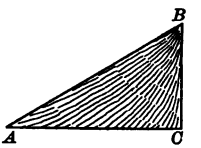
Solution. $BC = AC \cdot \tan A$.

We now measure the line AC and $\angle A$. The angle can be measured by a surveyor's instrument called the transit. Suppose $AC = 60$ feet and $\angle A = 49°$. From the table $\tan A = 1.15$.

Hence $BC = 60 \times 1.15 = 69$.

That is, the flagpole is 69 feet high.

Example 2. In the figure, A and B are two points on a hillside and it is required to find the horizontal distance AC and also the vertical distance BC.



Solution. Measure AB and $\angle A$. Suppose $AB = 245$ ft. and $\angle A = 37°$.

Then $BC = AB \cdot \sin 37° = 245 \times .602 = 147.49$,

and $AC = AB \cdot \cos A = 245 \times .799 = 195.755$.

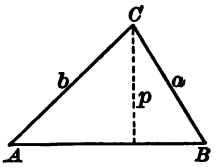
This sort of problem occurs very frequently in surveying land. Obviously it would be very difficult to measure directly either AC or BC.

EXERCISES

1. Find AB in the above figure if $CB = 40$ and $\angle A = 60°$.
2. Find AB if $AC = 84$ and $\angle A = 18°$.

THE RATIO OF THE SINES OF TWO ANGLES

384. THEOREM. *The ratio of the sides opposite two acute angles of a triangle is equal to the ratio of the sines of these angles.*



Given $\triangle ABC$ **with** $\angle A$ **and** $\angle B$ **both acute angles.**

To prove that $$\frac{\sin A}{\sin B} = \frac{a}{b}.$$

Proof: Draw the perpendicular p.

Then $$\sin A = \frac{p}{b} \text{ and } \sin B = \frac{p}{a}.$$

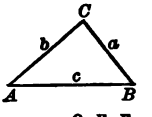
Hence, $$\frac{\sin A}{\sin B} = \frac{p}{b} \div \frac{p}{a} = \frac{a}{b}.$$ Q. E. D.

385. PROBLEM. *Given two angles and the included side of a triangle to find the other two sides.*

Solution. In the $\triangle ABC$ we suppose $\angle A$, $\angle B$, and side c are given. Then $\angle C$ may be obtained by subtracting the sum of $\angle A$ and $\angle B$ from 180°.

Then $$\frac{a}{c} = \frac{\sin A}{\sin C} \text{ or } a = \frac{c \sin A}{\sin C}.$$

And $$\frac{b}{c} = \frac{\sin B}{\sin C} \text{ or } b = \frac{c \sin B}{\sin C}.$$



Q. E. F.

EXERCISES

1. Find a and b if $\angle A = 64°$, $\angle B = 52°$ and $c = 46$.

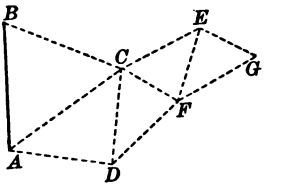
2. Find a and b if $\angle A = 80°$, $\angle B = 74°$, and $c = 192.5$.

This method is available only when among the parts given there is an angle and a side opposite it. Thus, we could not use this method to find parts of a triangle when two sides and the included angle are given.

INDIRECT MEASUREMENT

386. Surveying. The theorem of § 384 is of great importance in finding certain parts of a triangle when other parts are known. In land surveying on an extensive scale, processes similar to that used on the preceding page are constantly employed in finding the sides of triangles.

To begin with, a level piece of ground is selected and a line AB measured with great care. Then a point C is selected, and $\angle ABC$ and $\angle BAC$ measured very accurately with an instrument. Sides AC and BC may now be computed by means of the theorem, § 384, and a table of sines (see page 211). By measuring $\angle DAC$ and $\angle ACD$, CD and AD may be computed.



By this process, called triangulating, it is possible to survey a large territory without *directly measuring* any line except the first.

387. Labor Saving by Indirect Method. The saving of labor afforded by this *indirect method* of measuring is very great, and especially so in a rough and mountainous country, since measuring the straight line distance from one mountain peak to another by means of a measuring chain is *impossible.*

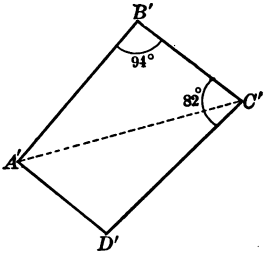
In practice, tables of logarithms are used and the sines are carried out to further decimal places, but the general process is as described above.

388. Distances to Inaccessible Objects. By this method, the distance to an inaccessible object may be measured with ease and accuracy. In the triangle ABC above, it is not at all necessary to be able to reach the point C, in order to find the distance from A to C. In a similar manner the distances from the earth to the moon, to the sun, and even to some of the fixed stars are measured. Of course, many refinements are introduced in making such measurements, but the general outline of the method is indicated above.

MAP MAKING

389. Map Making. After surveying a tract of land the surveyor makes a map of it. The map must be of the same shape as the tract it represents. That is, the tract and the map must be *similar*.

In measuring a four-sided figure denoted by $ABCD$ the surveyor finds that $AB = 97$ rods, $BC = 69$ rods, $CD = 100$ rods, $\angle B = 94°$, $\angle C = 82°$.*



To make the map let 1 inch represent 100 rods. Construct in order $A'B' = .97$ in., $\angle B' = 94°$, $B'C' = .69$ in., $\angle C' = 82°$, $C'D' = 1$ in. Draw $D'A'$.

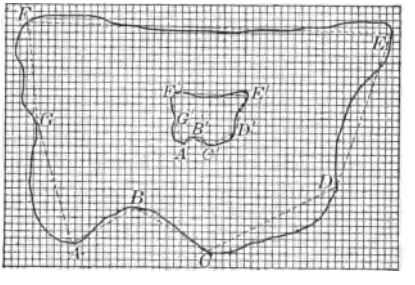
Prove that $A'B'C'D'$ is similar to $ABCD$.

Since 1 rod $= 16\frac{1}{2}$ feet,

100 rods $= 19{,}800$ inches.

Hence the ratio of similitude of $ABCD$ and $A'B'C'D'$ is $\frac{1}{19800}$. The map is said to be drawn to the scale 1 : 19800.

390. Mapping Irregular Figures. To map an irregular figure like $ABCDEFG$, construct a polygon $A'B'C'D'E'F'G'$ similar to the polygon $ABCDEFG$ and then sketch in the remaining parts of the boundary.



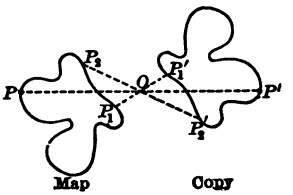
What is the scale of a map if one inch represents 30 rods?

* When AB, BC, CD and $\angle B$, $\angle C$ are determined, the whole figure is determined. Hence it is not necessary to measure DA, $\angle D$ or $\angle A$ unless it is done to check the other measurements.

MAP MAKING

391. Copying Maps. The following method is used to copy maps without changing their size:

Make a graduated ruler swing about a fixed point O as a pivot. Fasten the map on the drawing board and by means of the ruler construct a figure symmetrical to the given map with respect to the point O.

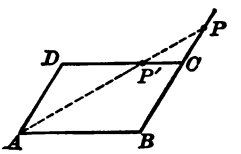


This involves the theorem: *If two figures are symmetrical with respect to a point they are equal.*

Prove this proposition for two triangles; also for two polygons in general.

392. To Enlarge or Reduce a Map. The following method is used to enlarge or reduce a map.

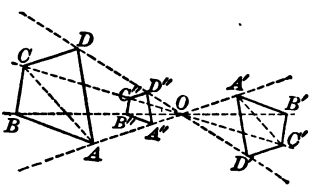
In the figure, $ABCD$ is a parallelogram whose sides are of constant length. The point A is fixed, while the remainder of the (solid line) figure is free to move. Show that the points P and P' trace out similar figures and that their ratio of similitude is $\frac{AP'}{AP} = \frac{BC}{BP}$.



This shows the essential parts of an instrument called the **pantograph**, which is much used by engravers to transfer figures and to increase or decrease their size. The point P is made to trace out the figure which is to be copied. Hence P' traces a figure similar to it. The scale or ratio of similitude is regulated by adjusting the length of CP.

This method is based on the following proposition:

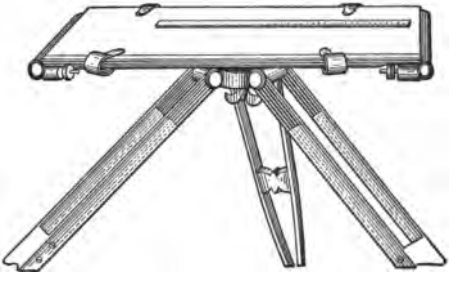
If two polygons have their corresponding vertices on lines which all pass through one point, and if they cut off proportional distances on these lines from the fixed point, then the polygons are similar.



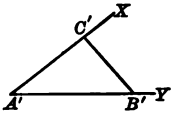
MAP MAKING

393. The Plane Table. The Plane Table consists essentially of a drawing-board mounted on a tripod. The board is provided with sights by means of which a ruler may be placed in any desired direction. Drawing paper is fastened to the board.

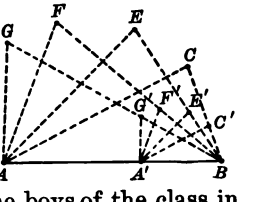
This table may be used in making maps of small areas where only moderate accuracy is required.



Drawing a map of a region may always be made to depend on the construction of triangles similar to triangles in the region to be mapped. Suppose we are to construct a triangle representing one whose vertices are three points A, B, C in the field. Take the plane table to the point A and by means of the ruler and sights draw lines $A'X$ and $A'Y$ in the directions AC and AB, respectively. Any point of $A'Y$ may be marked B'. Take the table to B and place it so that the line YA' passes through B and points toward A. Draw a line $B'C'$ in the direction BC, cutting $A'X$ at C'. The triangle $B'A'C'$ thus constructed is similar to the original triangle BAC. Why?



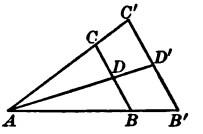
In the accompanying figure, C, E, F, G are points in the field as viewed from the points A and B, while C', E', F', G' are corresponding points on the map referred to A' and B.



A simple plane table can be made by the boys of the class in the manual training department of any high school.

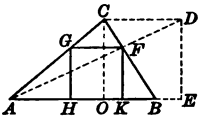
SIMILARITY OF FIGURES USED IN CONSTRUCTION

394. Problem 1. To construct a triangle having given two angles and the median m from one angle.



Construction. Construct any triangle ABC having the required angles and draw a median AD. Prolong AD to D', making $AD' = m$. Extend AB and AC and through D' draw $B'C' \parallel BC$, making $\triangle AB'C'$. Prove that $B'D' = D'C'$.

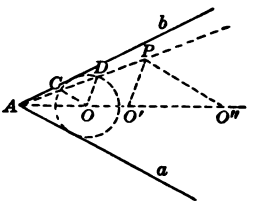
395. Problem 2. To inscribe a square in a given triangle.



Construction. Let ABC be the given triangle. Draw CD parallel to AB, making $CD = CO$. Complete the square $CDEO$.

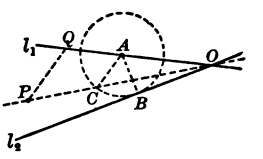
Draw AD meeting CB in F. Draw $FG \parallel AB$, $GH \parallel CO$, and $FK \parallel CO$. Then $FGHK \sim CDEO$ and hence is the square required.

396. Problem 3. To construct a circle through a given point and tangent to two given straight lines.



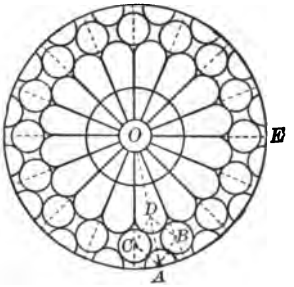
Construction. Let a and b be the given lines and P the given point. (1) Construct any circle O tangent to a and b. (2) Draw AP meeting the circle O in C and D. (3) Draw OC and OD and through P draw lines parallel to these meeting the bisector of the angle formed by a and b in O' and O''. (4) Prove that O' and O'' are centers of the required circles.

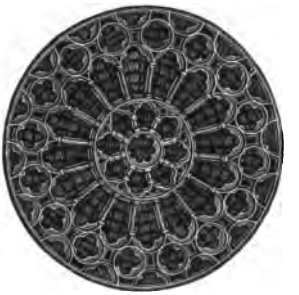
397. Problem 4. On a line to find a point equidistant from a given point and from a given line.



Construction. Let l_1 be the line on which the point is to be found, P the given point, and l_2 the other line. Let l_1 and l_2 meet in the point O. Construct a circle tangent to l_2 with the center at A, any point on l_1, and let the circle meet OP in C. Draw CA and $PQ \parallel CA$. Then Q is the required point. Prove. Discuss all possibilities.

Other Examples. Following are other instances of the use of this very important device in constructing figures that resist other methods of attack. It consists essentially in first constructing a figure similar to the one required and then constructing one similar to this and of the proper size.

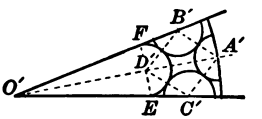




From Westminster Abbey.

398. Problem 5. Given a circle with radius OE. To construct within it the design shown in the figure.

Construction. Construct an angle O' equal to one sixteenth of a perigon. Through any point D' in the bisector of this angle draw a segment EF perpendicular to the bisector and terminated by the sides of the angle. On EF as a diameter construct a semicircle. Make $D'B' = EF = B'A' = D'C'$. Draw $C'A'$ and construct the small figure.



Now draw radii of the given circle OE dividing it into sixteen equal parts and bisect one of the central angles by a radius OA. Construct $\angle DAB = \angle D'A'B'$. Then AB is twice the radius of the required arcs and circles.

399. Problem 6. To construct a triangle similar to a given triangle $A'B'C'$ with its vertices on the sides of a given equilateral triangle ABC.

Suggestion. Construct any equilateral triangle $A''B''C''$ such that its sides pass through the points A', B', C'. Then $\triangle A''B''C'' \sim \triangle ABC$, and points may now be constructed on the sides of $\triangle ABC$ to form the vertices of the required triangle. Give the proof in full.

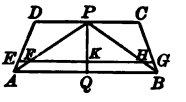
THEOREMS TO BE PROVED *

1. The line joining the middle points of two sides of a triangle is parallel to the third side. Compare § 184.

Prove by the methods of Book III.

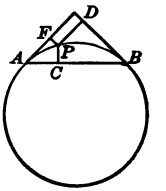
2. The square on the hypotenuse of a right triangle is equal to four times the square on the median to the hypotenuse.

3. A segment bisecting the two bases of a trapezoid bisects every segment which joins its other two sides and is parallel to the bases.



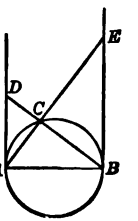
Prove $EF = HG$ and $FK = KH$, using § 340.

4. At the extremities of a chord AB tangents are drawn. From a point in the arc AB perpendiculars PC, PD, PF are drawn to the chord and the tangents. Prove that PC is a mean proportional between PD and PF.



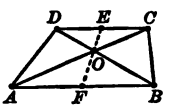
Suggestion. Draw chords AP and BP and prove $\triangle APF \sim \triangle BCP$ and $\triangle APC \sim \triangle BPD$.

5. In the figure AD and BE are tangents at the extremities of a diameter. If BD and AE meet in a point C on the circle prove that AB is a mean proportional between AD and BE.



6. If a triangle is inscribed in another by joining the middle points of the sides, then the inscribed triangle is similar to the original.

7. The line bisecting the bases of any trapezoid passes through the point of intersection of its diagonals.



Suggestion. Let E bisect DC, and draw EO meeting AB in F. Prove $AF = FB$.

* The supplementary matter on pages 200–219 is not required for the solution of any problem on pages 220–224.

PROBLEMS INVOLVING COMPUTATION

1. A point P is 16 inches from the center of a circle whose radius is 10 inches. Find the length of the tangent from P to the circle.

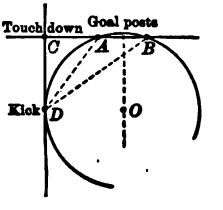
2. The length of a tangent from P to a circle is 10 inches, and the external segment of a secant is 6 inches. Find the length of the whole secant.

3. In two similar polygons two corresponding sides are 3 and 7. If the perimeter of the first polygon is 45, what is the perimeter of the second?

4. The perimeters of two similar polygons are 32 and 84. A side of the first is 11. What is the corresponding side of the second polygon?

5. The shortest distance from a point P to a circle is 8 inches and the greatest distance 20 inches. Find the length of the tangent from P.

6. The goal posts on a football field are $18\frac{1}{2}$ feet apart. If in making a touchdown the ball crosses the goal line 25 feet from the nearest goal post, how far back should it be carried so that the goal posts shall subtend the greatest possible angle from the place where the ball is placed? See Ex. 7, page 161, and § 358.



7. Solve the preceding problem if the touchdown is made a feet from the nearest goal post, and thus obtain a formula by means of which the distance may be computed in any case.

8. A building is 130 feet high, and a flagstaff on the top of it is 60 feet high; 130 feet from the base of the building in a horizontal plane, the flagstaff subtends a certain angle. How far from the building along the same line is there another point at which the staff subtends the same angle? At what distance does it subtend the same angle as it does at 300 feet?

Suggestion. Use § 359. Compare with Example 2, page 204.

PROBLEMS FOR ALGEBRAIC SOLUTION

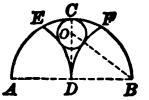
1. Find the altitude of an equilateral triangle whose side is 8; whose side is a.

2. If the altitude of an equilateral triangle is h, find its side.

3. Find the altitude on the longest side of a triangle whose sides are 6, 8, and 10.

4. In a circle of radius 10 a point P divides a chord into two segments 4 and 6. How far from the center is P?

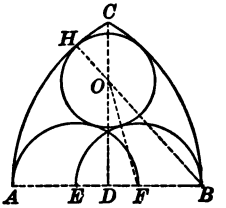
5. In the figure ACB is a semicircle. Arcs DE and DF are constructed with B and A as centers and radii equal to half AB. If $AD = 4$ feet, find the radius of $\odot OC$.



Suggestion. Show that the value of r is derived from the equation $(4 + r)^2 = (4 - r)^2 + 4^2$.

6. ABC is an equilateral arch. $AD = DB$, $AF = EB = \frac{2}{3}AB$. Semicircles are constructed with E and F as centers, and radius FB.

If $AB = s$, find OH and construct the figure.



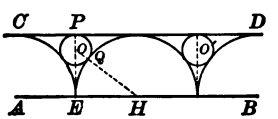
Suggestion. Let $OH = r$. Then $BO = s - r$, $FB = \frac{s}{3}$, $DF = \frac{s}{6}$, $FO = \frac{s}{3} + r$.

From $\triangle ODB$, $\overline{OD}^2 = \overline{OB}^2 - \overline{DB}^2 = (s - r)^2 - \left(\frac{s}{2}\right)^2$. (1)

From $\triangle ODF$, $\overline{OD}^2 = \overline{OF}^2 - \overline{DF}^2 = \left(\frac{s}{3} + r\right)^2 - \left(\frac{s}{6}\right)^2$. (2)

From (1) and (2) find r in terms of s.

7. Between two parallel lines a design is constructed as shown in the figure. If $HE = r$ and $OQ = r'$, find r' in terms of r.

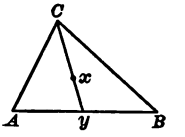


8. A segment of length 9 is divided into two segments whose ratio is $1 : \sqrt{2}$. Find the length of each.

PROBLEMS ON LOCI

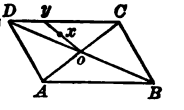
1. Find the locus of the middle points of the chords of a circle drawn through a fixed point within it.

2. If in the $\triangle ABC$ a segment Cy connects the vertex to any point y of the base, find the locus of the point x on this segment such that the ratio $Cx : Cy$ is the same for all points y.



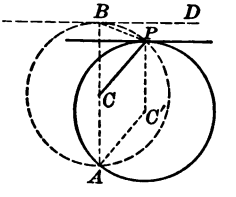
3. If on the base AB all $\triangle$ having equal altitudes are drawn, find the locus of the point x on the median Cy such that $Cx : Cy$ is the same for all the medians.

4. $ABCD$ is a $\square$ whose diagonals meet in O. If y is a point on any side of the $\square$, find the locus of a point x on the segment Oy such that $Oy : xy$ is the same for every such point y.



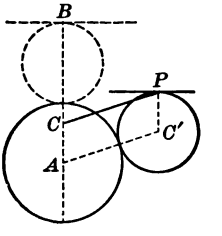
5. Through a fixed point on a circle chords are drawn and each extended to twice its length. Find the locus of the endpoints of these segments. Compare Example 2, p. 153.

6. A circle of constant radius passes through a fixed point A. A line tangent to it at the point P remains parallel to a fixed line BD. Find the locus of P.



Suggestion. Prove $AC'PC$ a parallelogram. Show that the locus consists of two circles.

7. The same as the preceding except that the circle of constant radius remains tangent to a fixed circle instead of passing through a fixed point.

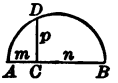


Suggestion. On that diameter of the fixed circle which is perpendicular to the fixed line lay off $AC = C'P$. Prove $AC'PC$ a parallelogram. Show that the locus consists of two circles.

8. Find the locus of a point P such that $PA \times PB$ is a constant, PA being a secant which meets the circle in A and B.

PROBLEMS ON CONSTRUCTION

1. If in the figure $m = 3$, $n = 5$, show that $p = \sqrt{3 \times 5} = \sqrt{15}$.



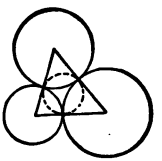
2. Show how to construct a segment $p = \sqrt{5}$; also $p = \sqrt{3}$; and $p = \sqrt{2}$.

3. Divide a line-segment 5 inches long into three parts proportional to 2, 3, and 4.

4. Divide a segment 11 inches long into parts proportional to 3, 5, 7, and 9.

5. Given a segment 8 inches long, to divide it into two parts proportional to 2 and $\sqrt{3}$. First construct a segment whose length is equal to $\sqrt{3}$.

6. Given three parallel lines l_1, l_2, l_3 and a triangle ABC. Construct a triangle $A'B'C'$ similar to ABC with its vertices on the lines l_1, l_2, l_3. Compare Example 3, page 157.



7. Construct circles with their centers at the vertices of a given triangle such that each circle is tangent to the other two.

8. On each of two sides of a given triangle ABC as chords construct arcs in which an angle of 120° may be inscribed. If these arcs meet in a point O inside the triangle, show that the three sides of the triangle subtend the same angle from the point O.

Problems 8, 9, 10 require the use of the supplementary problem in § 287.

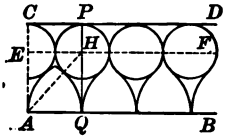
9. If $\angle 1 + \angle 2 + \angle 3 = 4$ rt. $\measuredangle$s, show how to find a point O within a given triangle ABC such that $\angle AOB = \angle 1$, $\angle BOC = \angle 2$, and $\angle COA = \angle 3$.

10. Given any two segments AB and CD and any two angles a and b. Find a point O such that $\angle AOB = \angle a$ and $\angle COD = \angle b$. Discuss the various possible cases and the number of points O in each case.

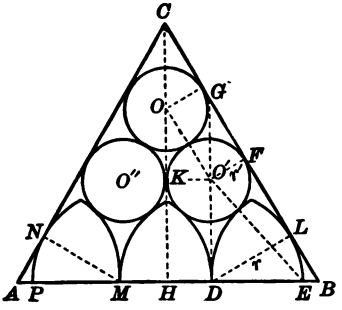
ARCS AND CIRCLES IN ARCHITECTURAL DESIGNS

The construction of the figures on this page requires computations based mainly on §§ 251, 254, 262, 263, and 350.

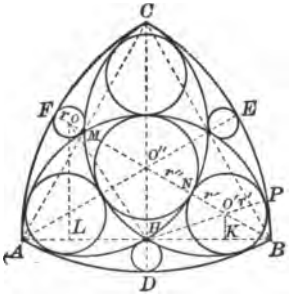
Ospedale Maggiore, Milan.



The Doge's Palace, Venice.



From Westminster Abbey.



From a Church in Chicago.

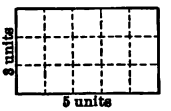
BOOK IV

AREAS OF POLYGONS

400. HERETOFORE certain properties of plane figures have been studied, such as *equality* and *similarity*, but no attempt has been made to measure the surface contained in such figures. For this purpose we first consider the rectangle.

401. Measuring a Rectangle. The surface of a rectangle is said to be *exactly measured* when we find how many times it contains a square unit.

E.g. if the base of a rectangle is five units long and its altitude is three units, its surface contains fifteen square units.



402. Area of a Rectangle. The number of times which a unit square is contained in a rectangle is called its **numerical measure,** or its *area.*

We distinguish three cases.

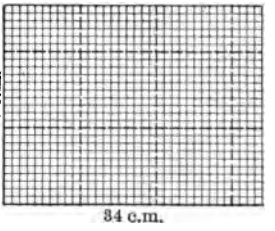
403. Case 1. If the sides of the given rectangle are *integral multiples* of the side of the unit square, then the area of the rectangle is determined by finding how many unit squares it contains.

Thus, if the sides of a rectangle are m and n units respectively, then it can be divided into m rows of unit squares, each row containing n squares. Hence the area of such a rectangle is $m \times n$ unit squares, that is, in this case

$$\text{area} = \text{base} \times \text{altitude}. \qquad (1)$$

404. Case 2. If the sides of the rectangle are *not integral multiples* of the side of the chosen unit square, but if the side of this square can be divided into equal parts such that the sides of the rectangle are integral multiples of one of these parts, then the area of the rectangle may be expressed *integrally* in terms of this smaller unit square, and *fractionally* in terms of the original unit.

For example, if the base is 3.4 decimeters and the altitude 2.6 decimeters, then the rectangle cannot be exactly divided into square decimeters, but it can be exactly divided into square centimeters. Each row contains 34 square centimeters and there are 26 such rows.



Hence, the area is $34 \times 26 = 884$ small squares or 8.84 square decimeters.

But $3.4 \times 2.6 = 8.84$. Hence, in this case also,

$$\text{area} = \text{base} \times \text{altitude}. \qquad (2)$$

405. Case 3. If a rectangle is such that there exists **no common measure whatever** of its base and altitude, then there is no square surface unit in terms of which its area can be *exactly* expressed. But by choosing a unit sufficiently small we may determine the area of a rectangle which differs as little as we please from the given rectangle.

E.g. if the base is 5 inches and the altitude is $\sqrt{5}$ inches, then the rectangle cannot be exactly divided into equal squares, however small.

But since $\sqrt{5} = 2.2361\cdots$, if we take as a unit of area a square whose side is one one-thousandth of an inch, then the rectangle whose base is 5 inches and whose altitude is 2.236 inches can be exactly measured as in cases 1 and 2, and its area is $5 \times 2.236 = 11.18$ square inches.

The small strip by which this rectangle differs from the given rectangle is less than .0002 of an inch in width, and its area is less than $5 \times .0002 = .001$ of a square inch.

By expressing $\sqrt{5}$ to further places of decimals and thus using smaller and smaller units of area, successive rectangles may be found which differ less and less from the given rectangle.

Similar reasoning would apply if the base were $\sqrt{7}$ inches and the altitude $\sqrt{5}$ inches.

An area thus obtained is called an **approximate area of the rectangle.**

In case the dimensions of the rectangle are incommensurable, then one or both can be measured only *approximately.*

We then have:

approximate area = approximate base × approximate altitude.

AREA OF A RECTANGLE

406. THEOREM I. *The area of a rectangle is equal to the product of its base and altitude.*

Proof. In case the sides of the rectangle are commensurable, the theorem is proved in cases (1) and (2), page 226.

If the sides are incommensurable, the argument used in case (3), page 227, shows that the theorem holds for all rectangles used in the process of approximation, however small the unit square may be. Hence, *the theorem applies to all practical measurements of the areas of rectangles.**

407. COROLLARY. *If two rectangles have equal bases, their areas are in the same ratio as their altitudes; and if they have equal altitudes their areas are in the same ratio as their bases.*

For if b, h and b', h' are the bases and altitudes, respectively, of the rectangles whose areas are A and A',

then, $$\frac{A}{A'} = \frac{bh}{b'h'}.$$

Hence, $$\frac{A}{A'} = \frac{h}{h'} \text{ if } b = b' \text{ and } \frac{A}{A'} = \frac{b}{b'} \text{ if } h = h'.$$

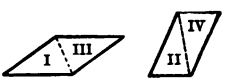
408. Areas of Polygons. From the formula for rectangles,

area = base × altitude,

we deduce formulas for the areas of other rectilinear figures by means of the following principle:

Two rectilinear figures are equivalent (that is, have the same area) *if they are equal* (that is, have the same shape and size), *or if they can be divided into parts which are equal in pairs.*

E.g. the two figures here shown are *equivalent* since ▲ I and III are equal respectively to II and IV, but the figures are not *equal*, since they are not of the same shape.

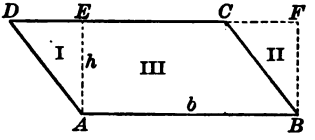


There are figures of equal area which cannot be thus divided. Thus a rectangle with base 2 and altitude 1 is equal in area to a square whose side is $\sqrt{2}$, though these figures cannot be divided in the manner stated above.

* For a theoretical proof see § 528 in the Appendix.

AREA OF A PARALLELOGRAM

409. THEOREM II. *The area of a parallelogram is equal to the product of its base and altitude.*



Given the $\square$ *ABCD* whose base is $AB = b$ and whose altitude is $AE = h$.

To prove that area $\square ABCD = bh$.

Proof: Draw $BF \perp DC$ (produced), forming the rectangle *ABFE*, whose base is *AB* and altitude *AE*.

In the rt. $\triangle$ I and II, $AD = BC$ and $AE = BF$. Why?

$\therefore \triangle \text{I} = \triangle \text{II}$. Why?

Now $\square ABCD$ is composed of $\triangle$ I and trapezoid III, while $\square ABFE$ is composed of $\triangle$ II and trapezoid III.

$\therefore$ Area $\square ABCD$ = area $\square ABFE$. § 408

(*Being composed of equal parts.*)

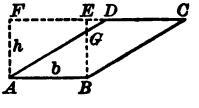
But area $\square ABFE = AB \times AE = bh$. § 406

$\therefore$ Area $\square ABCD = bh$. Q. E. D.

410. COROLLARY. *If two parallelograms have equal bases, their areas are in the same ratio as their altitudes; and if they have equal altitudes, their areas are in the same ratio as their bases.*

EXERCISE

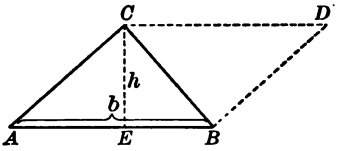
Prove the above theorem using the adjoining figure.



Suggestion. Show that $\square ABEF$ is composed of $\triangle ABG + \triangle AFD - \triangle EDG$, and that $\square ABCD$ is composed of the equal parts $\triangle ABG + \triangle BEC - \triangle EDG$.

AREA OF A TRIANGLE

411. THEOREM III. *The area of a triangle is equal to one half the product of its base and altitude.*



Given $\triangle ABC$ **with base** $AB = b$ **and altitude** $CE = h$.

To prove that $\text{area } \triangle ABC = \frac{1}{2}bh$.

Proof: Draw $CD \parallel AB$ and $BD \parallel AC$, thus forming the $\square ABDC$.

Then $\triangle ABC = \triangle BDC$. § 177, (1)

(*A diagonal divides a* $\square$ *into two equal* $\triangle$.)

$\therefore$ Area $\triangle ABC = \frac{1}{2}$ area $\square ABDC$.

But area $\square ABDC = bh$. § 409

$\therefore$ Area $\triangle ABC = \frac{1}{2}bh$. Q. E. D.

412. COROLLARY 1. *If two triangles have equal bases, their areas are in the same ratio as their altitudes; and if they have equal altitudes, their areas are in the same ratio as their bases.*

413. COROLLARY 2. *The product of the legs of a right triangle is equal to the product of the hypotenuse and the altitude upon the hypotenuse.*

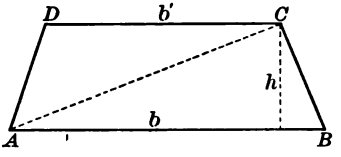
414. COROLLARY 3. *If a parallelogram and a triangle have equal bases and equal altitudes, the area of the triangle is one half that of the parallelogram.*

415. COROLLARY 4. *Two triangles have equal areas if they have the same base and if their vertices lie on a line parallel to the base.*

416. COROLLARY 5. *If* a_1, a_2, a_3 *are the sides of a triangle, and if* h_1, h_2, h_3 *are the corresponding altitudes upon them, then* $a_1 \times h_1 = a_2 \times h_2 = a_3 \times h_3$.

AREA OF A TRAPEZOID

417. THEOREM IV. *The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.*



Given the trapezoid *ABCD* with bases *b* and *b'* and altitude *h*.

To prove that area $ABCD = \frac{1}{2}h(b + b')$.

Proof: Draw the diagonal AC, forming the two $\triangle ABC$ and ACD with the common altitude h.

Then area $\triangle ABC$ + area $\triangle ACD = \frac{1}{2}bh + \frac{1}{2}b'h$. § 411

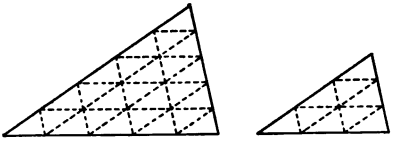
$\therefore$ Area $ABCD = \frac{1}{2}bh + \frac{1}{2}b'h = \frac{1}{2}h(b + b')$. Q. E. D.

418. COROLLARY. *The area of a trapezoid is equal to the product of its altitude and its median.*

Suggestion. Use § 187.

EXPERIMENTAL GEOMETRY

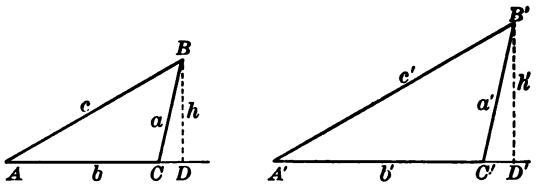
1. In the figure, the two triangles are similar and their ratio of similitude is 5 : 3. Count the small equal triangles to find the ratio of their areas. How does the ratio of the areas correspond to the ratio of the squares of the lengths of the corresponding sides?



2. Construct a triangle ABC and divide each of its sides into four equal parts. Then construct a triangle $A'B'C'$ such that $A'B'$ contains seven of the four equal parts of AB, $B'C$ contains seven of the equal parts of BC, etc. Connect division points in these triangles as in Example 1 and count the small triangles to compare the areas of $\triangle ABC$, $A'B'C'$.

RATIO OF THE AREAS OF SIMILAR TRIANGLES

419. THEOREM V. *The areas of similar triangles are in the same ratio as the squares of any two corresponding sides, or as the squares of any two corresponding altitudes.*



Given similar △ ABC and $A'B'C'$, with altitudes h and h'.

To prove that

$$\frac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{c^2}{c'^2} = \frac{h^2}{h'^2}.$$

Proof: $\dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \dfrac{\frac{1}{2}bh}{\frac{1}{2}b'h'} = \dfrac{b}{b'} \times \dfrac{h}{h'}.$ § 411

But $\dfrac{h}{h'} = \dfrac{b}{b'}.$ § 341

(*In similar* ▲ *corresponding altitudes are in the same ratio, etc.*)

$\therefore \dfrac{b}{b'} \times \dfrac{h}{h'} = \dfrac{b^2}{b'^2} = \dfrac{h^2}{h'^2}.$ Why?

$\therefore \dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \dfrac{b^2}{b'^2} = \dfrac{h^2}{h'^2}.$ Why?

But $\dfrac{b}{b'} = \dfrac{a}{a'} = \dfrac{c}{c'}.$ § 334, (2)

(*Similar polygons have their corresponding sides proportional.*)

$\therefore \dfrac{b^2}{b'^2} = \dfrac{a^2}{a'^2} = \dfrac{c^2}{c'^2},$ § 312, (7).

and $\dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \dfrac{a^2}{a'^2} = \dfrac{b^2}{b'^2} = \dfrac{c^2}{c'^2} = \dfrac{h^2}{h'^2}.$ Why?

Q. E. D.

SIGHT WORK

1. Find the area of a parallelogram whose base and altitude are 8 and 7, respectively.

2. The area of a parallelogram is 63 and its base is 9. Find its altitude.

3. The area of a parallelogram is 48 and its altitude is 6. Find its base.

4. Find the area of a triangle whose base and altitude are 10 and 6, respectively.

5. The bases of a trapezoid are 8 and 6 and its altitude is 5. Find the area.

6. Corresponding sides of two similar triangles are 3 and 5. What is the ratio of their areas?

7. The ratio of similitude of two similar triangles is 1 : 3. The area of the smaller is 25. Find the area of the larger.

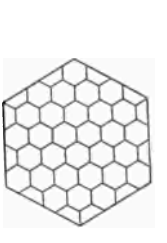
8. The areas of two similar triangles are 25 and 144. Find their ratio of similitude.

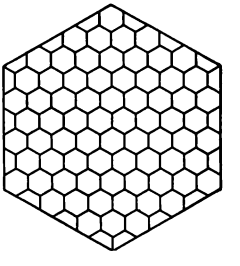
EXPERIMENTAL GEOMETRY

1. Construct similar rectangles $ABCD$ and $A'B'C'D'$, making the dimensions of $ABCD$ 3 inches and 5 inches, and those of $A'B'C'D'$ 6 inches and 10 inches.

How does the ratio of the areas correspond to the ratio of the squares of the corresponding sides?

2. The figures to the right are similar regular hexagons. It may easily be verified that the ratio of similitude is 6 : 9 by using one side of a small hexagon in each figure as a unit, and noticing that the diagonal of a regular hexagon is twice its side. Count the small hexagons inside each figure. This may be done most easily by drawing the diagonals and thus dividing each hexagon into six equal parts.



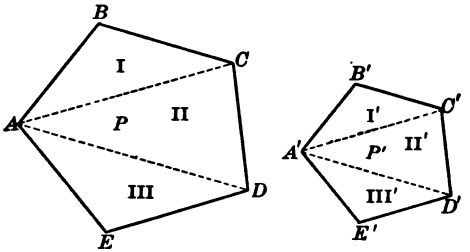


Compare the ratio of the areas with the square of the ratio of similitude.

Compare these experiments with that on similar triangles on page 231 and with the theorem on page 232.

RATIO OF THE AREAS OF SIMILAR POLYGONS

420. THEOREM VI. *The areas of two similar polygons are in the same ratio as the squares of any two corresponding sides.*



Given two similar polygons whose areas are P and P'.

To prove that $\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \cdots.$

Proof: Let I, I′; II, II′; III, III′ represent the areas of the corresponding similar triangles formed by drawing the diagonals from A and A'. § 363

Then $\frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\text{I}}{\text{I}'} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\text{II}}{\text{II}'} = \frac{\overline{AD}^2}{\overline{A'D'}^2} = \frac{\text{III}}{\text{III}'}.$ § 419

(*The areas of similar* ▲ *are in the same ratio as the squares, etc.*)

That is, $\frac{\text{I}}{\text{I}'} = \frac{\text{II}}{\text{II}'} = \frac{\text{III}}{\text{III}'}.$

$\therefore \frac{P}{P'} = \frac{\text{I} + \text{II} + \text{III}}{\text{I}' + \text{II}' + \text{III}'} = \frac{\text{I}}{\text{I}'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$ § 318

But $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \cdots.$ Why?

$\therefore \frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \cdots.$ § 312, (7)

Q. E. D.

421. COROLLARY. *The areas of two similar polygons are in the same ratio as the squares of any two corresponding line-segments.*

SIGHT WORK

1. The ratio of similitude of two similar triangles is $\sqrt{3} : \sqrt{5}$. What is the ratio of their areas?

2. Corresponding sides of a series of similar polygons may be represented by 1, 2, 3, 4, 5, 6. If 1 represents the area of the first polygon, what numbers will represent the areas of the others?

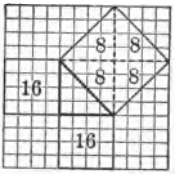
3. The area of the smaller of two similar polygons is 9 and their ratio of similitude is 1 : 4. Find the area of the larger polygon.

4. The ratio of the areas of two similar polygons is 3 : 8. A side of the smaller is 3 inches long. Find the length of the corresponding side of the larger.

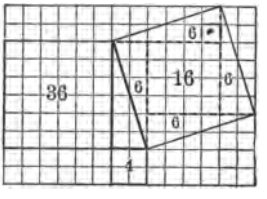
5. The ratio of the perimeters of two similar polygons is 3 : 11, and the area of the smaller polygon is 74. Find the area of the larger polygon.

EXPERIMENTAL GEOMETRY

1. In the first figure the heavy triangle is an isosceles right triangle. Squares are constructed on the hypotenuse and on the two sides. Show by counting small squares that the square constructed on the hypotenuse is equal in area to the sum of the squares on the two legs.



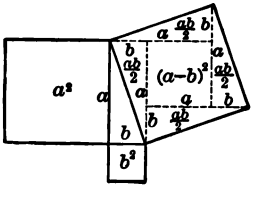
2. In the second figure the right triangle is scalene. Show that the square on the hypotenuse equals 40 of the small squares. Note that here also the square on the hypotenuse is equal in area to the sum of the squares on the other two sides.



3. Using the third figure, show that the same theorem holds for any right triangle.

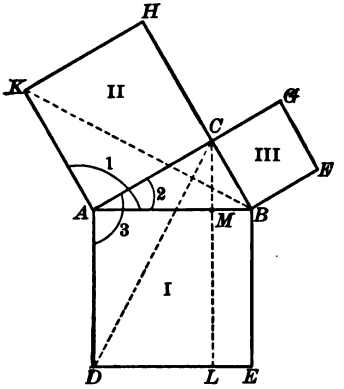
Hint. Call the legs of the given triangle a and b. Tell how the dotted lines are drawn and give the explanation in full.

This is the Pythagorean theorem which was proved in § 350 and will be proved again in § 422. The proof just given is due to Bhâskara (born 1114 A.D.), who constructed the figure and simply said "Behold!".



THE PYTHAGOREAN THEOREM

422. THEOREM VII. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*



Given the rt. △ *ABC* with squares I, II, and III on the hypotenuse and two sides respectively.

To prove that *area* □ *I* = *area* □ *II* + *area* □ *III.*

Proof: Draw *CD* and *BK* and let fall *CL* ⊥ *DE.*

In ≜ *ABK* and *ACD*, ∠1 = ∠3.

(*Since each equals a rt.* ∠ + ∠2.)

Also *AC* = *AK* and *AB* = *AD.* Given.

∴ △ *ABK* = △ *ACD.* Why?

Now area △ *ACD* = ½ area □ *ADLM.* § 414

(*Having the same base AD and same altitude DL.*)

Also △ *ABK* and □ II have the same base *AK* and same altitude *AC.*

(*Since* ∠ *ACH* = ∠ *ACB* = *rt.* ∠, *therefore HCB is a straight line.*)

Hence Area △ *ABK* = ½ area □ II. Why?

∴ area □ II = area □ *ADLM.* Why?

Pythagoras (569–500 B.C.), born on the Island of Samos, was the first of the great Greek mathematicians. He studied in Egypt, where no doubt he learned the practical geometry of the Egyptians. Later he returned to Samos to teach, but soon migrated westward to Sicily, and finally settled in the Greek colony of Croton in Southern Italy.

Ball, the historian, says of him: "It was Pythagoras who gave to geometry that rigorous character of deduction which it still bears, and made it the foundation of a liberal education; and there is good reason to believe that he was the first to arrange the leading propositions of the subject in logical order."

In like manner by drawing CE and AF, we may show that

area □ III = area ▭ $LEBM$.

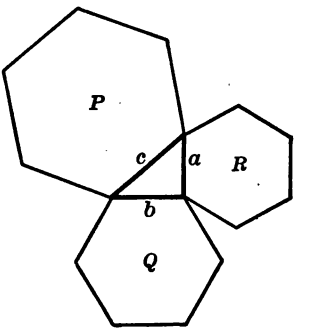
But area ▭ $ADLM$ + area ▭ $LEBM$ = area □ I.

∴ area □ I = area □ II + area □ III. Q. E. D.

423. COROLLARY 1. *The square on either leg of a right triangle is equivalent to the square on the hypotenuse minus the square on the other leg.*

424. COROLLARY 2. *The area of a square is equal to one half the area of the square on its diagonal.*

425. COROLLARY 3. *If similar polygons are constructed on the hypotenuse and legs of a right triangle, then the polygon on the hypotenuse is equivalent to the sum of the other two polygons.*



Suggestion. $\frac{R}{P}=\frac{a^2}{c^2}$ and $\frac{Q}{P}=\frac{b^2}{c^2}$. § 420

(*The areas of two similar polygons are in the same ratio as the squares etc.*)

$$\therefore \frac{R}{P}+\frac{Q}{P}=\frac{a^2}{c^2}+\frac{b^2}{c^2}=\frac{a^2+b^2}{c^2}=\frac{c^2}{c^2}=1,$$

and $$\frac{R+Q}{P}=1 \text{ or } R+Q=P.$$

HISTORICAL NOTE

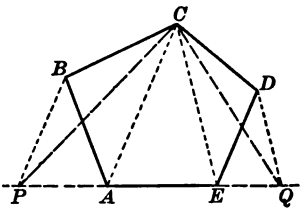
The Pythagorean Theorem. The theorem of § 422 is one of the most famous in all mathematics. It is now fairly certain that the general theorem was first stated and proved by Pythagoras, though the story that he sacrificed 100 oxen to the gods on the occasion may be questioned. Special cases of the theorem were known to the Egyptians as early as 2000 B.C., as for example, that a triangle whose sides are 3, 4, 5 is right-angled. Such special cases were also know to the Chinese and Babylonians at an early date.

In this connection the Pythagoreans also discovered the irrational number, that is, that there are numbers such as $\sqrt{2}$ which cannot be expressed exactly as integers or as ordinary fractions.

It is not known what proof Pythagoras used for his theorem, though it was probably similar to the one given in § 350 of this text. The proof given in § 422 is the one used by Euclid. Many other forms of proof have been worked out.

REDUCING A POLYGON TO A TRIANGLE

426. PROBLEM. *To construct a triangle equivalent to a given polygon.*



Given the polygon *ABCDE*.

To construct $\triangle PCQ$ *equivalent to* $ABCDE$.

Construction and proof. Draw AC and CE. Draw $BP \parallel AC$ and $DQ \parallel CE$ to meet EA extended at P and Q. Draw CP and CQ.

Then $\text{area } \triangle APC = \text{area } \triangle ABC$. § 415

Hence $PCDE$ is equivalent to $ABCDE$, since

$$\text{area } PCDE = \text{area } ACDE + \text{area } APC \text{ and}$$
$$\text{area } ABCDE = \text{area } ACDE + \text{area } ABC.$$

In like manner we may show that $\text{area } PCQ = \text{area } PCDE$.

Hence $\triangle PCQ$ is equivalent to polygon $ABCDE$. Q. E. F.

EXERCISES

1. Draw a line from a vertex of a triangle to a point in the opposite side which shall divide the triangle into two triangles whose areas are in the ratio 2 : 1.

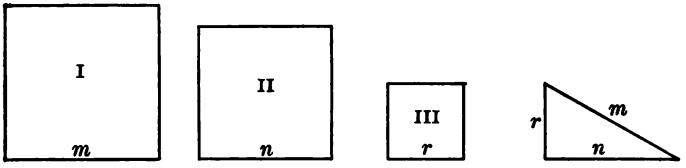
2. Divide a parallelogram into three equivalent parts by lines drawn from one vertex. Use the construction in Ex. 1.

3. The sides of two equilateral triangles are 8 and 6 respectively. Find the side of an equilateral triangle whose area is equal to the sum of their areas.

4. Corresponding sides of two similar triangles are a and b. Find the side of a third triangle similar to these whose area is equal to the sum of their areas.

ADDING AND SUBTRACTING SQUARES

427. PROBLEM. *To construct a square equivalent to the sum or difference of two given squares.*



(1) **Given squares II and III with sides n and r.**

To construct square I such that area I = area II + area III.

Construction. Construct a rt. △ with legs n and r equal to the sides of squares II and III respectively.

Then square I with the side m equal to the hypotenuse of the triangle is the square sought. Why?

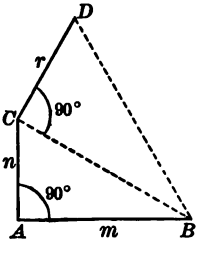
(2) **Given squares I and II.**

To construct a square III such that area III = area I − area II.

Construction. Construct a rt. △ with hypotenuse m equal to a side of square I, and with a leg n equal to a side of square II.

Then square III with side r equal to the other leg of the triangle is the square sought. Why?

Q. E. F.



428. COROLLARY. *To construct a square equivalent to the sum of three squares whose sides are m, n, r.*

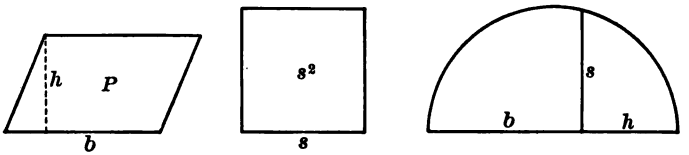
The square on BC equals $m^2 + n^2$ and the square on BD equals $(m^2+n^2) + r^2 = m^2 + n^2 + r^2$.

EXERCISE

Find a side of a square field which contains the same area as three city lots each 10 rods square.

SQUARE EQUIVALENT TO A PARALLELOGRAM

429. PROBLEM. *To construct a square equivalent to a given parallelogram.*



Given the parallelogram P.

To construct a square s^2 equivalent to P.

Construction. Upon a diameter $b + h$, equal to the sum of the base and altitude of P, construct a semicircle.

At the division point between b and h erect $s \perp$ diameter.

Then the square erected on the side s is the square required. § 348

(*If from a point on a ⊙ a ⊥ is drawn to a diameter, etc.*)

Complete the proof. Q. E. F.

430. COROLLARY 1. *To construct a square equivalent to a given triangle.*

Suggestion. Use $h + \frac{1}{2}b$ for the diameter.

431. COROLLARY 2. *To construct a square equivalent to a given polygon.*

Suggestion. First reduce the polygon to a triangle. § 426

EXERCISES

1. Using the method § 429, construct a square whose area shall be twice that of a given square.

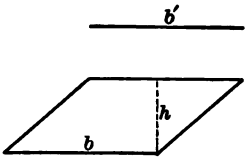
2. Construct a square whose area shall be three times that of a given square.

3. Construct a parallelogram equivalent to a given square s^2.

Suggestion. With any diameter greater than $2s$ find a point in the circumference from which the perpendicular to the diameter is equal to s.

RECTANGLE EQUIVALENT TO A PARALLELOGRAM

432. PROBLEM. *To construct a rectangle on a given base and equivalent to a given parallelogram.*



Given b and h the base and altitude of the given parallelogram, b' the base of the required rectangle, and x the unknown altitude.

Construction. We are to determine x so that $b'x = bh$; that is, so that $b' : b = h : x$.

Hence, x is the fourth proportional to b', b, and h. Construct this fourth proportional, showing the complete solution. (See § 328.) Q. E. F.

This construction is attributed to Pythagoras. It represents a much higher achievement than the discovery of the Pythagorean proposition itself.

433. COROLLARY. *To construct a rectangle on a given base and equivalent to a given triangle.*

EXERCISES

1. Construct a rectangle on a given base equivalent to a given irregular quadrilateral. First use § 426.

2. Construct a rectangle on a given base equivalent to an irregular hexagon.

3. On a side of a regular hexagon as a base construct a rectangle equivalent to the hexagon.

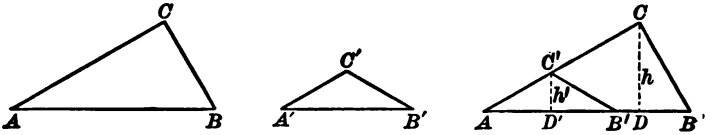
4. Construct a parallelogram on a given base equivalent to a given triangle. Is there more than one solution?

5. Construct a square whose area shall be four times that of a given square; five times; one half the area; one fifth.

6. Construct an isosceles triangle, with a given altitude h, equivalent to a given triangle.

SUPPLEMENTARY THEOREMS AND PROBLEMS *

434. THEOREM VIII. *If an angle of one triangle is equal to an angle of another, their areas are in the same ratio as the products of the sides including the equal angles.*



Given $\triangle ABC$ and $A'B'C'$ with $\angle A = \angle A'$.

To prove that $\dfrac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \dfrac{AB \cdot AC}{A'B' \cdot A'C'}$.

Proof: Represent area $\triangle ABC$ by I and area $\triangle A'B'C'$ by II. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$, and draw the altitudes h and h'.

Then $\text{I} = \frac{1}{2} h \cdot AB$ and $\text{II} = \frac{1}{2} h' \cdot A'B'$. Why?

$$\therefore \frac{\text{I}}{\text{II}} = \frac{\frac{1}{2} h \cdot AB}{\frac{1}{2} h' \cdot A'B'} = \frac{h}{h'} \cdot \frac{AB}{A'B'}.$$ Why?

But $h' \parallel h$. § 149

(*Two lines* $\perp$ *to the same line, etc.*)

Then $\triangle AC'D' \sim \triangle ACD$ and therefore

$$\frac{h}{h'} = \frac{AC}{A'C'}.$$ § 341

$$\therefore \frac{\text{I}}{\text{II}} = \frac{AC}{A'C'} \cdot \frac{AB}{A'B'} = \frac{AB \cdot AC}{A'B' \cdot A'C'}.$$ Why?

Q. E. D.

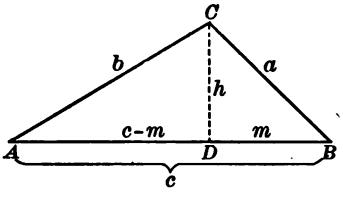
SIGHT WORK

Find the ratio of the areas of two triangles ABC and $A'B'C$ if $\angle A = \angle A'$, $AC = 8$, $AB = 10$, $A'C' = 6$, and $A'B' = 7$.

* Pages 242–249 contain supplementary matter some or all of which may be omitted without destroying the continuity.

AREA OF A TRIANGLE IN TERMS OF ITS SIDES

435. PROBLEM. *To express the area of a triangle in terms of its three sides.*



Solution. The area of $\triangle ABC = \frac{1}{2} AB \times CD = \frac{1}{2} hc$.

We first find m in terms of a, b, c.

We have $h^2 = a^2 - m^2$ and $h^2 = b^2 - (c - m)^2$. § 423

$$\therefore a^2 - m^2 = b^2 - c^2 + 2cm - m^2,$$

and
$$m = \frac{a^2 + c^2 - b^2}{2c}.$$

Hence
$$h^2 = a^2 - \left(\frac{a^2 + c^2 - b^2}{2c}\right)^2 = \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4c^2}$$

$$= \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4c^2}$$

$$= \frac{[(a+c)^2 - b^2][b^2 - (a-c)^2]}{4c^2}$$

$$= \frac{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}{4c^2}.$$

Now call $a + b + c = 2s$, or $a + c - b = 2s - 2b$.

Then $a + c - b = 2(s - b)$, $b + a - c = 2(s - c)$,

and $b + c - a = 2(s - a)$.

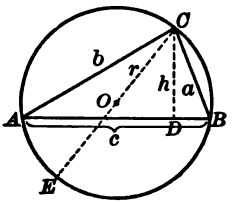
Hence
$$h^2 = \frac{2s \cdot 2(s-b) \cdot 2(s-c) \cdot 2(s-a)}{4c^2},$$

or
$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore \text{Area } \triangle ABC = \tfrac{1}{2} hc = \sqrt{s(s-a)(s-b)(s-c)}.$$ Q. E. F.

AREA IN TERMS OF RADIUS OF CIRCUMSCRIBED CIRCLE

436. PROBLEM. *To express the area of a triangle in terms of the three sides and the radius of the circumscribed circle.*



Solution. In the figure $CE = 2r$, where r is the radius of the circumscribed circle.

Then $ab = 2r \cdot h.$ § 380

(*The product of two sides of a* $\triangle$ *is equal to the product of the altitude from the vertex in which these sides meet and the diameter of the circumscribed* $\odot$.)

But $h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$ § 435

Hence $ab = \frac{4r}{c}\sqrt{s(s-a)(s-b)(s-c)},$

or $\frac{abc}{4r} = \sqrt{s(s-a)(s-b)(s-c)}.$

But area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}.$ § 435

$\therefore$ area of $\triangle ABC = \frac{abc}{4r}.$ Q. E. F.

EXERCISES

1. The sides of a triangle are a, b, c. Find the radius of the circumscribed circle, using the results obtained in § 436.

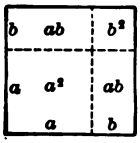
2. Find the radius of a circle circumscribed about a triangle whose sides are 8, 10, and 12. Give the result correct to two places of decimals.

GEOMETRIC CONSTRUCTION OF ALGEBRAIC FORMULAS

437. PROBLEM. *To construct geometrically the formula:*

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Construction. Let a and b represent the numerical measures of two line-segments. Then, the square erected upon the segment $a+b$ may be subdivided as shown in the figure, giving

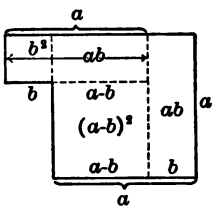


$$(a+b)^2 = a^2 + 2ab + b^2.$$ Q. E. F.

438. PROBLEM. *To construct geometrically the algebraic formula:*

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Construction. In this case the whole figure consists of the square on the segment a, together with the square on the segment b added at the upper left corner.



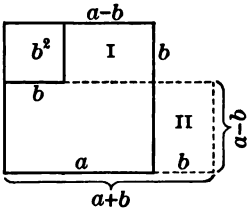
When the two rectangles ab are subtracted from the whole figure, there is left the square on the segment $a-b$. That is,

$$a^2 + b^2 - 2ab = (a-b)^2.$$ Q. E. F.

439. PROBLEM. *To construct geometrically the algebraic formula:*

$$(a+b)(a-b) = a^2 - b^2.$$

Construction. In the upper left corner of the square on a construct the square on b. Draw dotted lines forming the rectangles I and II, each having the dimensions b and $a-b$.

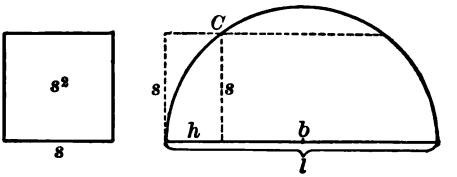


Then the rectangle with base $a+b$ and altitude $a-b$ is equivalent to $a^2 + \text{II} - \text{I} - b^2$ or $a^2 - b^2$. That is,

$$(a+b)(a-b) = a^2 - b^2.$$ Q. E. F.

GEOMETRIC SOLUTION OF THE EQUATIONS $\begin{cases} bh = s^2 \\ b + h = l \end{cases}$

440. PROBLEM. *To find two line-segments, b and h, such that $b + h$ equals a given segment l, and such that the rectangle bh equals a given square s^2.*



Given the square s^2 and the segment l.

To divide l into two parts b and h, such that $bh = s^2$.

Construction. On l as a diameter draw a semicircle.

At one end of l erect a perpendicular equal to s and through the top of it draw a line parallel to l cutting the circle at C.

From C let fall a perpendicular to l.

This perpendicular is equal to s. Why?

Then h and b are the segments sought.

For $b + h = l$ and $bh = s^2$. Why?

Q. E. F.

441. COROLLARY. *To construct a parallelogram equivalent to a given square and having the sum of its base and altitude equal to a given line-segment.*

EXERCISES

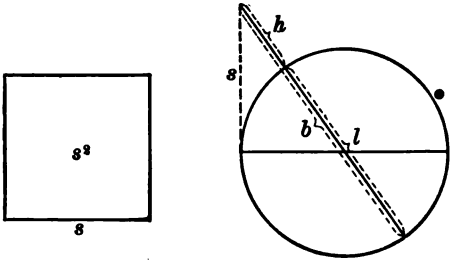
1. Construct a rectangle whose area is 36 square inches and whose perimeter is 26 inches. (The sum of base and altitude is 13 inches.)

2. Construct a rectangle whose area is 32 square inches and whose perimeter is 20 inches.

Suggestion. First construct a segment whose length is $\sqrt{32}$.

GEOMETRIC SOLUTION OF THE EQUATIONS $\begin{cases} bh = s^2 \\ b - h = l \end{cases}$

442. PROBLEM. *To find two line-segments, b and h, such that $b - h$ equals a given segment l, and such that the rectangle bh equals a given square s^2.*



Given the square s^2 and the segment l.

To find two segments b and h such that $b - h = l$ and $bh = s^2$.

Construction. On l as a diameter draw a circle. At one end of l erect a perpendicular equal to s.

From the upper extremity draw a secant through the center of the circle, terminating in the circle.

Then the whole secant is the segment b sought and the external segment is h.

For $b - h = l$ and $bh = s^2$. § 358

Q. E. F.

443. COROLLARY. *To construct a parallelogram equivalent to a given square and having the difference of its base and altitude equal to a given line-segment.*

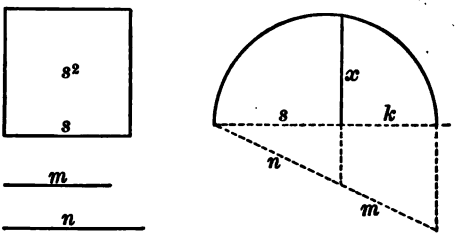
EXERCISES

1. Construct a rectangle whose area is 49 square inches and whose length exceeds its width by 3 inches.

2. Construct a rectangle whose area is 27 square inches and whose length exceeds its width by 1 inch.

SQUARES IN A GIVEN RATIO

444. PROBLEM. *To construct a square having a given ratio $m:n$ to a given square.*



Given the square s^2 and segments m and n.

To construct a square x^2 such that $x^2 : s^2 = m : n$.

Construction. Construct k a fourth proportional to s, n, and m as in § 328.

Upon the segment $s + k$ as a diameter construct a semicircle. At the division point between s and k erect a perpendicular x. Then the square on x as a side is such that $x^2 : s^2 :: m : n$.

Proof:

$x^2 = ks$. Why?

Hence, dividing both sides by s^2,

$$\frac{x^2}{s^2} = \frac{k}{s}.$$

But $\frac{k}{s} = \frac{m}{n}$. Why?

$$\therefore \frac{x^2}{s^2} = \frac{m}{n}.$$ Why?

Q. E. F.

445. COROLLARY. *To construct a polygon similar to a given polygon and having a given ratio, $m:n$, to it.*

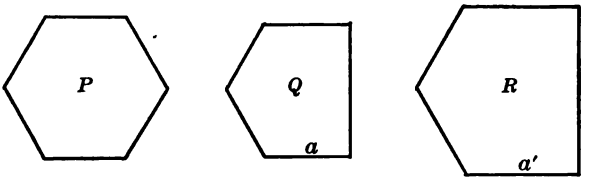
Suggestion. If s is a side of the given polygon, find s' such that $\frac{s'^2}{s^2} = \frac{m}{n}$.

EXERCISE

Given a square whose side is s. Construct a square whose area is to s^2 as $3:2$. Compare Examples 1 and 2, page 240.

EQUIVALENT AND SIMILAR POLYGONS

446. PROBLEM. *To construct a polygon similar to a given polygon and equivalent to another given polygon.*



Given polygons P and Q.

To construct a polygon R similar to Q and equivalent to P.

Construction. Reduce P and Q to equivalent triangles; § 426
and reduce these triangles to equivalent squares. § 430

Call the sides of these squares p and q respectively.

Now find a segment a' such that

$$p : q = a' : a. \qquad \text{§ 328}$$

Construct polygon R similar to Q on the side a' corresponding to a.

Then polygon R is the required polygon.

Proof: Since $p : q = a' : a,$

we have $p^2 : q^2 = a'^2 : a^2.$ § 312, (7)

But p^2 and q^2 are the areas of P and Q respectively,

and $R : Q = a'^2 : a^2.$

$\therefore P : Q = R : Q.$ Why?

That is, R is equivalent to P. § 315

Q. E. F.

EXERCISES

1. Construct an equilateral triangle whose area is equal to that of a square whose side is 4.

2. Construct a triangle whose sides are proportional to 2, 3, and 4 inches, and whose area is 36 square inches.

3. Construct a regular hexagon whose area is 25 square inches.

EXERCISES IN COMPUTATION*

1. Given two concentric circles whose radii are r and r'. Find the length of a chord of the greater which is tangent to the smaller.

2. Show that the altitude of an equilateral triangle with side s is $\frac{s}{2}\sqrt{3}$.

3. If two equal circles of radius r intersect so that each passes through the center of the other, find the length of the common chord.

4. In a right triangle the hypotenuse is 10 feet and the difference between the other sides is 2 feet. Find the sides.

5. The sides of two equilateral triangles are a and b respectively. Find the side of an equilateral triangle whose area is equal to the sum of their areas.

6. Draw a line parallel to the base of a triangle and cutting two of its sides. How will the resulting triangle and trapezoid compare in area, (a) If each of the two sides of the given triangle is bisected? (b) If each of the two sides of the new triangle is three times the corresponding side of the trapezoid?

Suggestion. First compare the two triangles using § 419, then use § 312, (5).

7. The sides of a triangle are 6, 8, 9. A line parallel to the longest side divides the triangle into a trapezoid and a triangle of equal areas. Find the ratio in which the line divides the two sides.

8. Draw a line parallel to the base of any triangle, and cutting two of its sides. How do the altitudes of the resulting triangle and trapezoid compare, (a) If they are equal in area? (b) If the area of the new triangle is three times that of the trapezoid?

* The supplementary matter, pages 242–249, is not required for any of the examples on pages 250–254.

THEOREMS TO BE PROVED

1. The area of a rhombus is one half the product of its diagonals.

2. The area of an isosceles right triangle is equal to the square on the altitude let fall upon the hypotenuse.

3. If the angles of a triangle are 30°, 60°, 90°, and if the side opposite the 60° angle is a, find the other sides.

4. If in a parallelogram $ABCD$, any point E on the diagonal BD is joined to A and C, then $\triangle BEA$ and BEC are equivalent, and also $\triangle DEA$ and DEC are equivalent.

5. If E is any point within a parallelogram $ABCD$, then $\triangle ABE + \triangle CDE$ equals one half the area of the parallelogram.

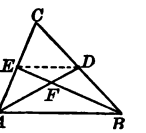
6. If the middle points of the sides of a quadrilateral are connected, then a parallelogram is formed whose area is half the area of the quadrilateral.

7. The square on a diagonal of a rectangle is equal to half the sum of the squares on the diagonals of the squares constructed on two adjacent sides of the rectangle.

8. The diagonals of a trapezoid form with the non-parallel sides two triangles which have equal areas.

Suggestion. Make use of Example 7, page 220. Then prove that in the figure of that example area $AFED$ = area $FBCE$, area AFO = area FBO, area DEO = area EOC.

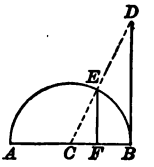
9. If AD and BE are medians of the triangle ABC, AD meeting BE in F, then the area of $\triangle ABF$ equals the area of the quadrilateral $CDFE$.



10. If the middle points of adjacent sides of any parallelogram are joined, the four triangles thus formed are equal in area.

PROBLEMS IN CONSTRUCTION

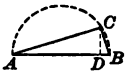
1. Inscribe a square in a given semicircle.



Construction. Let AB be the diameter of the given semicircle. At B construct a perpendicular to AB, making $BD = AB$. Connect D with the center C, meeting the circle at E. Let fall EF perpendicular to AB. Then EF is a side of the required square. Complete the figure and make the proof by showing that $EF = 2\ CF$.

2. Inscribe a square in a semicircle and another in a quadrant of the same circle. Compare their areas.

3. On a given line-segment AB as a hypotenuse construct a right triangle such that the altitude upon the hypotenuse shall meet it at a given point D.



4. By means of Example 3, and § 347, construct two segments HK and LM such that the ratio of the squares on these segments shall equal a given ratio.

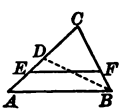
5. On a given segment AB find a point D such that $\frac{\overline{AB}^2}{\overline{AD}^2} = \frac{2}{1}$.

6. Divide a given segment into two segments such that the areas of the squares constructed upon them shall be in a given ratio.

7. Construct a line parallel to the base of a triangle such that the resulting triangle and trapezoid shall be equivalent.

Suggestion. Draw the altitude and find a point on it as in Example 5. Prove that the parallel through this point is the one sought.

Another solution of Example 7 is as follows:



Analysis. Suppose EF is the required line; draw the median BD. Since $\triangle BCD$ is equivalent to one half the original triangle, it follows that $\triangle CEF$ and CDB have equal areas. Hence, $CE \times CF = CD \times CB$ (§ 434), *i.e.* $CD : CE = CF : CB = CE : CA$. Therefore, CE is a mean proportional between CD and CA. (This alternative solution depends on one of the supplementary theorems of this chapter.)

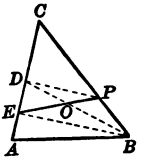
PROBLEMS IN CONSTRUCTION

8. Draw a line parallel to the base of a triangle so that the resulting trapezoid shall contain one-nth of the area of the triangle.

Suggestion. Suppose the altitude CD of the given $\triangle$ is divided into two parts, CE and ED, so that $\overline{CE}^2 : \overline{ED}^2 = n - 1 : 1$. Then the $\triangle$ cut off by a line $\parallel$ to the base through E will contain $(n - 1)n$ths of the given $\triangle$.

Construct a figure accurately for $n = 3$.

9. Through a point on a side of a triangle draw a line dividing it into two equivalent parts.

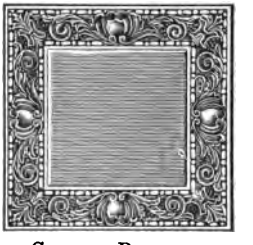


Construction. Let P be the given point. Draw the median BD. Draw $BE \parallel PD$ and draw PE. This is the required line. Prove.

Suggestion. Notice that area $\triangle DPE$ = area $\triangle DPB$ by § 415, and hence that area $\triangle CEP$ = area $\triangle CBD$.

10. Through a given point on a side of a triangle draw a line which divides it into two figures whose areas are in the ratio $\frac{1}{3}$.

11. A border is to be constructed about a given square with an area equal to one half that of the square. By geometrical construction find the outer side of the border if the side of the square is given.



CEILING PATTERN

Suggestion. The ratio of the areas of the given square and of the large square is $1 : \frac{3}{2}$. Let s_1 and s_2 be sides of any two squares whose areas are in the ratio $1 : \frac{3}{2}$ (see Example 4), and let s_3 be the side of the given small square. Then find s_4 so that $s_1 : s_2 = s_3 : s_4$.

12. A border is to be constructed about a given regular octagon, such that its area is equal to that of the octagon. If a side of the given octagon is a given segment AB, find by geometrical construction a segment equal to an outer side of the border.

CEILING PATTERN

MISCELLANEOUS EXERCISES

1. Given two lines at right angles to each other. Find the locus of all points such that the sum of the squares of the distances from the lines is 25.

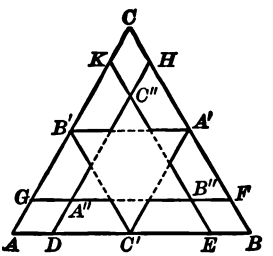
2. Two equal circles move so as to remain tangent to each other and also tangent respectively to two lines perpendicular to each other. Find the locus of the points of tangency of the two circles.

Suggestion. Note that the point of tangency bisects their line of centers and that the centers move along lines which are parallel to the given lines and hence at right angles to each other.

3. If tangents are drawn to a circle at the extremities of a diameter and if another line tangent to the circle at P meets these two tangents in A and B respectively, show that $AP \cdot PB = r^2$, where r is the radius of the circle.

Suggestion. If C is the center of the circle, show that $\angle ACB$ is a right angle. Then apply § 345 (1).

4. A', B', C' are the middle points of the sides of the equilateral triangle ABC. The sides of the triangle $A'B'C'$ are trisected and segments drawn as shown in the figure.



FROM CHURCH OF OR SAN MICHELE

(*a*) Prove that $A''B''C''$ is an equilateral triangle.

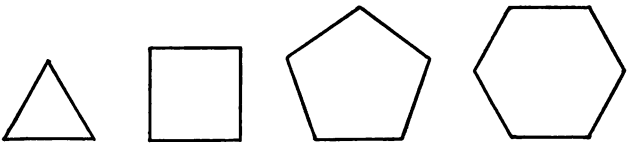
(*b*) If the area of $\triangle ABC = a$, find the area of $\triangle A''B''C''$.

(*c*) Construct the whole figure showing that the conditions given fix the figure.

BOOK V

REGULAR POLYGONS AND CIRCLES

447. Regular Polygons. A *regular polygon* is one which is both equilateral and equiangular.



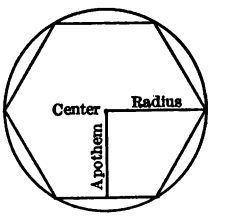
REGULAR POLYGONS

According to this definition, determine whether each of the following polygons is *regular* or not and state why:

An equilateral triangle, an equiangular triangle, a rectangle, a square, a rhombus. Draw a figure to illustrate each. Draw a triangle which fulfills neither condition of the definition, also a quadrilateral.

It is proved in §§ 453, 454 that a circle may be circumscribed about, and also inscribed in, a regular polygon, and that these circles have the same center. See §§ 250, 280.

448. Center. The *center of a regular polygon* is the common center of its inscribed and circumscribed circles.



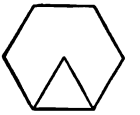
449. Radius. The *radius of a regular polygon* is the distance from the center to one of its vertices.

450. Apothem. The *apothem of a regular polygon* is the perpendicular distance from its center to a side.

Clearly, the apothem is the radius of the inscribed circle.

REGULAR POLYGONS

451. Angle at the Center. The angle formed by joining two adjacent vertices of a regular polygon to its center is called the *angle at the center* or the *central angle* of the polygon.



452. Construction of Regular Polygons. The general problem of constructing a regular polygon depends upon the division of a circle into as many equal parts as the polygon has sides.

The problem of dividing the circle into equal parts can be solved in some cases by the methods of elementary geometry, and some of these methods will be considered in this book. In most cases this problem cannot be solved by elementary methods.

E.g. the circle may be divided into 2, 3, 4, 5, 6, 8, 10, 12, 15, equal parts, but not into 7, 9, 11, 13, 14, equal parts by these methods.

If a circle has already been divided into a certain number of equal parts, it may then be divided into twice, four times, eight times, etc., that number of parts by repeated bisection of the arcs.

The division of the circle into equal parts depends upon the theorem (§ 244) that equal central angles intercept equal arcs on the circle, and hence it involves the subdivision of angles into equal parts.

SIGHT WORK

1. What is the angle at the center of a regular triangle? a regular quadrilateral? a pentagon? a hexagon?

2. How may the angle at the center of any regular polygon be found?

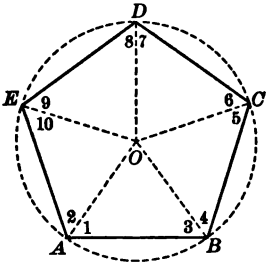
3. Show how a regular hexagon may be formed by placing six regular triangles together about a point.

4. How many degrees in each angle of a regular triangle? of a regular quadrilateral? a pentagon? a hexagon?

5. How may the number of degrees in an angle of any regular polygon be found?

CIRCLE CIRCUMSCRIBED ABOUT A REGULAR POLYGON

453. THEOREM I. *If a polygon is regular, a circle may be circumscribed about it.*



Given a regular polygon *ABCDE*.

To prove that a point O can be found such that

$$AO = BO = CO = DO = EO.$$

Construction. Bisect $\angle A$ and $\angle B$ and let the bisectors meet in a point O. Then O is the center of the required circle.

Outline of proof. Give reasons for each of the following:

(1) $\angle 1 = \angle 3$, (2) $AO = BO$,

(3) $\triangle ABO = \triangle CBO$, (4) $AO = OC$ and $\angle 1 = \angle 5$.

(5) Since $\angle A = \angle C$ and $\angle 1 = \angle 5$. $\therefore \angle 2 = \angle 6$.

(6) $\angle 5 = \angle 6$. (7) $\triangle BCO = \triangle DCO$.

Continuing this process prove $\triangle OCD = \triangle EDO$, etc.

$$\therefore AO = BO = CO = DO = EO.$$

454. COROLLARY 1. *If a polygon is regular, a circle concentric with the circumscribed circle may be inscribed in it.*

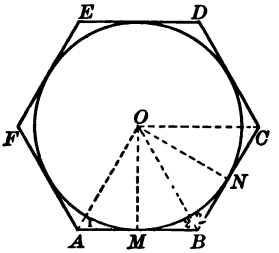
For the sides are equal chords of the circumscribed circle and hence are equally distant from the center of that circle. The same point is, then, the center of both the inscribed and circumscribed circles.

455. COROLLARY 2. *An equilateral polygon inscribed in a circle is regular.*

Suggestion. Show that the angles are measured by halves of equal arcs.

EQUIANGULAR POLYGON CIRCUMSCRIBED ABOUT A CIRCLE

456. THEOREM II. *If an equiangular polygon is circumscribed about a circle, then the polygon is regular.*



Given polygon *ABCDEF* with $\angle A = \angle B = \angle C$, etc., and circumscribed about $\odot O$.

To prove that $AB = BC = CD$, etc.

Proof: Draw the radii *OA*, *OB*, *OC*, etc., to the vertices, and *OM*, *ON*, etc., to the points of tangency.

Then $\angle 2 = \angle 3$. § 278

(*Two tangents to a* $\odot$ *from a point outside are equal, etc.*)

$\therefore \angle 1 = \angle 2$, being halves of the equal $\measuredangle A$ and *B*.

Now $\triangle AMO = \triangle BMO$. § 166

$\therefore AM = BM$, and similarly $BN = CN$.

But $BM = BN$. § 278

$\therefore AB = BC$. Why?

By joining the center to the other vertices and points of tangency, we prove similarly that

$$BC = CD = DE = EF = FA.$$

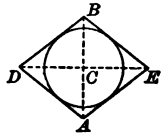
$\therefore ABCDEF$ is a regular polygon. § 447

Q. E. D.

457. COROLLARY. *Tangents drawn at the vertices of a regular inscribed polygon form a regular circumscribed polygon.*

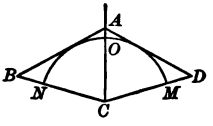
EXERCISES ON EQUILATERAL CIRCUMSCRIBED POLYGONS

1. Points A and B are taken on a diameter extended such that $AC = BC$. Segments AD, BD, AE, BE are tangents meeting in D and E on the diameter perpendicular to AB. Prove $AD = BD = BE = AE$.



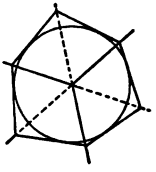
Note. — It is clear that the angles of this circumscribed quadrilateral are equal only in case $AC = DC$. Since AC need not be equal to DC, it follows that an *equilateral* circumscribed polygon may not be *equiangular*. Compare the theorem of § 456.

2. In the figure, C is the center of the arc MON, and $\angle DCA = \angle BCA$. AD and AB are tangents to arcs OM and ON drawn from any point A in CO extended. Prove $AB = AD$.



Suggestion. In $\triangle ACD$ and ACB we have $\angle CAB = \angle CAD$. § 278

3. Problem. Construct an equilateral circumscribed polygon of six sides which shall not be equiangular.

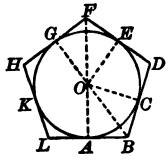


Suggestion. Divide the circle into three equal parts, by making angles of 120° at the center. In connection with each arc construct a figure like that in Example 2. This will form the required polygon. Note that this construction leads always to a polygon with an *even* number of sides.

4. If an equilateral polygon of an *odd* number of sides is circumscribed about a circle, it is equiangular.

Suggestion. $BC + CD = DE + EF$. $\therefore BC = EF$ since $DC = DE$. Likewise we prove $AB = GF$. Now prove $\angle B = \angle F$ by using $\triangle OBC$ and OEF and § 278. Alternate angles are thus proved equal.

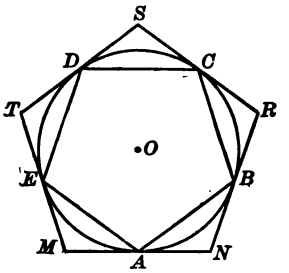
If the number of angles is *odd*, this proves that all the angles are equal. For, suppose the angles are 1, 2, 3, 4, 5, 6, 7. Then the above proof shows that $\angle 1 = \angle 3 = \angle 5 = \angle 7 = \angle 2 = \angle 4 = \angle 6$. Notice that if the number of angles were even, we should get two sets of angles *equal among themselves* but we should not know whether those of one set are equal to those of the other.



458. THEOREM III. *If a circle is divided into any number of equal parts, then*

(1) *the chords joining the division points, taken in order, form a regular polygon; and*

(2) *the tangents at the division points, taken in order, form a regular polygon.*



Given a ⊙ divided into equal parts by points *A*, *B*, *C*, *D*, *E* with chords and tangents drawn as shown in the figure.

To prove that (1) *ABCDE is a regular polygon, and* (2) *MNRST is a regular polygon.*

Proof: (1) Chords *AB*, *BC*, *CD*, etc., are equal. § 247 (2)

$\therefore$ *ABCDE* is a regular polygon. § 455

(*An equilateral polygon inscribed in a ⊙ is regular.*)

(2) This follows immediately from § 457. Q. E. D.

459. COROLLARY 1. *If a regular polygon is inscribed in a circle, a regular inscribed polygon with double the number of sides will be formed by joining the midpoints of the equal arcs to the vertices of the given polygon.*

460. COROLLARY 2. *If a regular polygon is circumscribed about a circle, a regular circumscribed polygon with double the number of sides will be formed by drawing tangents at the midpoints of the equal arcs.*

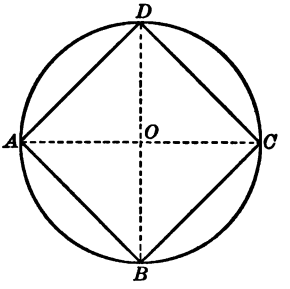
461. Uses of Circumscribed and Inscribed Polygons. An important use of inscribed and circumscribed polygons is in connection with finding the circumference and the area of a circle. (See pages 266 and following.)

If the processes indicated in Corollaries 1 and 2 on page 260 be repeated indefinitely, the perimeters and the areas of the inscribed polygons will continually increase and those of the circumscribed polygons will continually decrease.

This is easily seen by using the fact that the sum of two sides of a triangle is greater than the third side.

INSCRIBING A SQUARE IN A CIRCLE

462. PROBLEM. *To inscribe a square in a circle.*



Construction. Through the center O draw two diameters perpendicular to each other, meeting the circle in the points A, B, C, and D.

Then $ABCD$ is the square required.

Proof: Show that $ABCD$ is both equilateral and equiangular. Q. E. F.

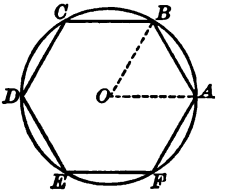
463. COROLLARY 1. *To construct a circumscribed square.*

464. COROLLARY 2. *To construct a regular inscribed polygon of eight sides, of sixteen sides, etc.*

465. COROLLARY 3. *To construct a regular circumscribed polygon of eight sides, of sixteen sides, etc.*

INSCRIBING A REGULAR HEXAGON

466. PROBLEM. *To inscribe a regular hexagon in a circle.*



Construction. Draw a chord AB equal to the radius OA, and draw the radii OA and OB.

Then $\triangle AOB$ is equilateral and therefore equiangular.

$$\therefore \angle AOB = 60° = \tfrac{1}{6} \text{ of } 360°.$$

That is, arc AB is $\frac{1}{6}$ of the circle.

Then if with the compasses, using the chord AB as a radius, chords AB, BC, CD, DE, EF, FA are stepped off, these chords will intercept equal arcs of 60° each, and the last division point will fall on A.

Then $ABCDEF$ is a regular hexagon. § 455

(*An equilateral polygon inscribed in a* ⊙, *etc.*)

Q. E. F.

467. COROLLARY 1. *To circumscribe a regular hexagon.*

Suggestion. Find the vertices of the regular inscribed hexagon and draw tangents at these vertices. See § 458 (2).

468. COROLLARY 2. *To inscribe and circumscribe regular polygons of twelve, twenty-four, forty-eight sides, etc.*

Suggestion. To construct the circumscribed polygons, use the method suggested in Corollary 1.

469. COROLLARY 3. *To inscribe a regular triangle.*

Suggestion. Draw chords connecting alternate vertices of the regular hexagon, as A, C, and E.

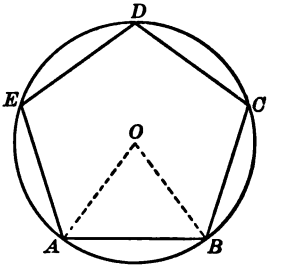
470. COROLLARY 4. *To circumscribe a regular triangle.*

REGULAR POLYGONS OF FIVE, TEN, AND FIFTEEN SIDES *

471. I. *To inscribe in a circle a regular pentagon.*

Each central angle is 72°. Why?

Using § 372, construct at O a central angle of 72°, intercepting the chord AB.



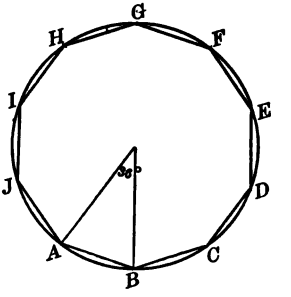
Using AB as a side, step off five equal chords, AB, BC, CD, etc., each intercepting an arc of 72°. Hence the last division point will fall on A.

Then $ABCDE$ is a regular pentagon. Why?

472. II. *To inscribe in a circle a regular decagon.*

Each central angle is 36°. Why?

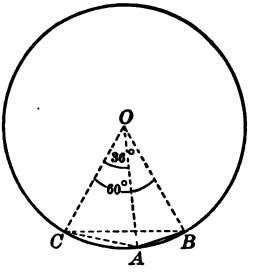
Either bisect the central angles of the regular pentagon just found, or using § 372, construct at O an angle of 36°, intercepting the chord AB. Then with AB as a side, step off ten equal chords as before.



Then $ABCDEFGHIJ$ is a regular decagon. Why?

473. III. *To inscribe in a circle a regular fifteen-sided polygon.*

Let CB be the side of a regular hexagon, and CA the side of a regular decagon.



Then $\overset{\frown}{CB} = 60°$ and $\overset{\frown}{CA} = 36°$.

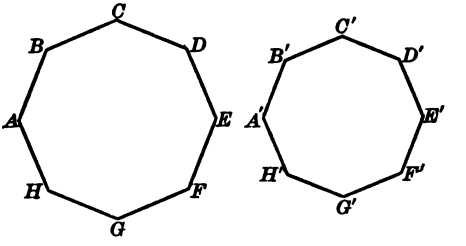
$\therefore \overset{\frown}{AB} = 60° - 36° = 24° = \frac{1}{15}$ of 360°.

Then AB is a side of the regular inscribed polygon of 15 sides.

* This page may be omitted without destroying the continuity. If taken, then also § 372 must be studied.

SIMILAR REGULAR POLYGONS

474. THEOREM IV. *Two regular polygons of the same number of sides are similar.*



Given polygons *ABCDEFGH* and *A'B'C'D'E'F'G'H'* both regular and each having *n* sides.

To prove that they are similar.

Proof:

$\angle A = \angle A'$, $\angle B = \angle B'$, etc., since each is $\frac{2n-4}{n}$ rt. $\measuredangle$s.

(*The sum of the $\measuredangle$s of a polygon having n sides is, etc.*) § 194

$AB = BC = CD$, etc., and $A'B' = B'C' = C'D'$, etc. § 447

$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EF}{E'F'} \cdots$. Why?

Hence the two polygons are similar. § 334

Q. E. F.

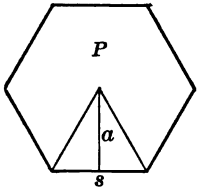
475. COROLLARY 1. *The ratio of the radii and also of the apothems of two regular polygons of the same number of sides is equal to the ratio of similitude of the two polygons.*

476. COROLLARY 2. *The perimeters of two regular polygons of the same number of sides are in the same ratio as their radii or as their apothems.*

477. COROLLARY 3. *The areas of two regular polygons of the same number of sides are in the same ratio as the squares of their radii, and also as the squares of their apothems.*

AREA OF A REGULAR POLYGON

478. THEOREM V. *The area of a regular polygon is equal to one half the product of its apothem and perimeter.*



Given a regular polygon P with apothem a, perimeter p, and side s.

To prove that area $P = \frac{1}{2}ap$.

Proof: The polygon is divided by its radii into equal isosceles triangles each with altitude a and base s. Why?

The area of one such triangle is $\frac{1}{2}as$. Why?

$$\therefore \text{area } P = \tfrac{1}{2}a(s + s + s + \cdots) = \tfrac{1}{2}ap. \quad \text{Q. E. D.}$$

479. COROLLARY. *The area of any polygon circumscribed about a circle is equal to one half the product of the perimeter of the polygon and the radius of the circle.*

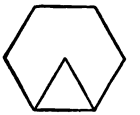
SIGHT WORK

1. A polygon whose perimeter is 48 inches is circumscribed about a circle of radius 4 inches. Find the area of the polygon.

2. The radii of two regular polygons of the same number of sides are 3 and 7 inches. Find the ratio of their perimeters. Also find the ratio of their areas.

3. Regular polygons having the same number of sides are circumscribed about circles whose radii are 2 and 6 inches. Find the ratio of the perimeters of these polygons and also of their areas.

REGULAR POLYGONS



451. Angle at the Center. The angle formed by joining two adjacent vertices of a regular polygon to its center is called the *angle at the center* or the *central angle* of the polygon.

452. Construction of Regular Polygons. The general problem of constructing a regular polygon depends upon the division of a circle into as many equal parts as the polygon has sides.

The problem of dividing the circle into equal parts can be solved in some cases by the methods of elementary geometry, and some of these methods will be considered in this book. In most cases this problem cannot be solved by elementary methods.

E.g. the circle may be divided into 2, 3, 4, 5, 6, 8, 10, 12, 15, equal parts, but not into 7, 9, 11, 13, 14, equal parts by these methods.

If a circle has already been divided into a certain number of equal parts, it may then be divided into twice, four times, eight times, etc., that number of parts by repeated bisection of the arcs.

The division of the circle into equal parts depends upon the theorem (§ 244) that equal central angles intercept equal arcs on the circle, and hence it involves the subdivision of angles into equal parts.

SIGHT WORK

1. What is the angle at the center of a regular triangle? a regular quadrilateral? a pentagon? a hexagon?

2. How may the angle at the center of any regular polygon be found?

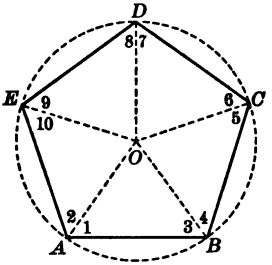
3. Show how a regular hexagon may be formed by placing six regular triangles together about a point.

4. How many degrees in each angle of a regular triangle? of a regular quadrilateral? a pentagon? a hexagon?

5. How may the number of degrees in an angle of any regular polygon be found?

CIRCLE CIRCUMSCRIBED ABOUT A REGULAR POLYGON

453. THEOREM I. *If a polygon is regular, a circle may be circumscribed about it.*



Given a regular polygon *ABCDE*.

To prove that a point O can be found such that

$$AO = BO = CO = DO = EO.$$

Construction. Bisect $\angle A$ and $\angle B$ and let the bisectors meet in a point O. Then O is the center of the required circle.

Outline of proof. Give reasons for each of the following:

(1) $\angle 1 = \angle 3$, (2) $AO = BO$,

(3) $\triangle ABO = \triangle CBO$, (4) $AO = OC$ and $\angle 1 = \angle 5$.

(5) Since $\angle A = \angle C$ and $\angle 1 = \angle 5$. $\therefore \angle 2 = \angle 6$.

(6) $\angle 5 = \angle 6$. (7) $\triangle BCO = \triangle DCO$.

Continuing this process prove $\triangle OCD = \triangle EDO$, etc.

$$\therefore AO = BO = CO = DO = EO.$$

454. COROLLARY 1. *If a polygon is regular, a circle concentric with the circumscribed circle may be inscribed in it.*

For the sides are equal chords of the circumscribed circle and hence are equally distant from the center of that circle. The same point is, then, the center of both the inscribed and circumscribed circles.

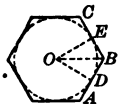
455. COROLLARY 2. *An equilateral polygon inscribed in a circle is regular.*

Suggestion. Show that the angles are measured by halves of equal arcs.

EXERCISES

1. Is every equiangular polygon inscribed in a circle regular? Does it depend on whether the number of sides is even or odd?

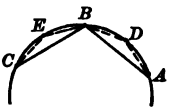
Compare the examples on page 259.



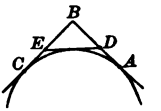
2. Show that the radius of a regular polygon bisects the angle at the vertex to which it is drawn.

Suggestion. Inscribe a circle in the polygon.

3. Show that the perimeter of a regular polygon of a given number of sides is less than that of one having twice the number of sides, both being inscribed in the same circle.



4. Show that the perimeter of a regular polygon is greater than that of one having twice the number of sides, both being circumscribed about the same circle.



Suggestion. Show that $AB + BC > AD + DE + EC$.

5. Compare the areas of the two polygons in Example **3**.

6. Compare the areas of the two polygons in Example **4**.

7. Show that the area of a square inscribed in a circle of radius r is $2r^2$. How does this compare with the area of the circumscribed square?

8. Compute the apothem and area of a regular inscribed hexagon if the radius of the circle is r; also of the regular circumscribed hexagon.

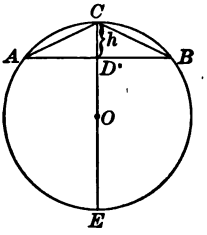
9. A regular triangle is inscribed in a circle of radius **10**. Find the apothem and a side of the triangle.

10. Find the locus of the vertices of all regular polygons of the same number of sides which can be circumscribed about the same circle.

11. Find the locus of the middle points of the sides of all regular polygons of the same number of sides which are inscribed in the same circle.

SIDES OF SUCCESSIVE INSCRIBED POLYGONS

486. PROBLEM. *To compute a side of a regular inscribed polygon of $2n$ sides in terms of a side of a regular inscribed polygon of n sides.*



Given ⊙ O in which AB is a side of a regular inscribed polygon of n sides.

To find the length of AC, a side of a regular inscribed polygon of $2n$ sides.

Solution. $AD = DB$ and $\angle ADC =$ rt. $\angle$.

Denote AB by S_n and AC by S_{2n}.

Then $(S_{2n})^2 = \overline{AC}^2 = \overline{AD}^2 + h^2 = (\frac{1}{2}S_n)^2 + h^2$. (1)

But $h \times DE = h(2r - h) = AD \times DB = \overline{AD}^2$. § 354

That is, $\overline{AD}^2 = (\frac{1}{2}S_n)^2 = h(2r - h)$. (2)

Solving equation (2) for h, we have

$$h = \frac{2r \pm \sqrt{4r^2 - S_n^2}}{2}.$$

Taking the negative sign, since $h < r$, squaring and transposing, $h^2 + (\frac{1}{2}S_n)^2 = 2r^2 - r\sqrt{4r^2 - S_n^2}$.

Hence, from (1), $(S_{2n})^2 = 2r^2 - r\sqrt{4r^2 - S_n^2}$,

and $S_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - S_n^2}}$ (3)

Q. E. F.

487. COROLLARY. *If the radius of the circle is unity, then*

$$S_{2n} = \sqrt{2 - \sqrt{4 - S_n^2}}.$$

PERIMETERS OF INSCRIBED AND CIRCUMSCRIBED POLYGONS

488. If the first inscribed polygon in § 486 is a regular hexagon, and if $r = 1$, then $S_6 = 1$.

In this case, substituting in the formula of § 487, we have,

$$S_{12} = \sqrt{2 - \sqrt{4 - 1}} = 0.51763809 \cdots.$$

Denoting the perimeter of a regular inscribed polygon of n sides by P_n, we have,

$$P_{12} = 12(0.51763809) = 6.21165708 \cdots.$$

Computing S_{24}, P_{24}, S_{48}, P_{48}, etc., in a similar manner, we have,

$S_{12} = \sqrt{2-\sqrt{4-1}} = .51763809 \quad \therefore P_{12} = 6.21165708$

$S_{24} = \sqrt{2-\sqrt{4-(.51763809)^2}} = .26105238 \quad \therefore P_{24} = 6.26525722$

$S_{48} = \sqrt{2-\sqrt{4-(.26105238)^2}} = .13080626 \quad \therefore P_{48} = 6.27870041$

$S_{96} = \sqrt{2-\sqrt{4-(.13080626)^2}} = .06543817 \quad \therefore P_{96} = 6.28206396$

$S_{192} = \sqrt{2-\sqrt{4-(.06543817)^2}} = .03272346 \quad \therefore P_{192} = 6.28290510$

$S_{384} = \sqrt{2-\sqrt{4-(.03272346)^2}} = .01636228 \quad \therefore P_{384} = 6.28311544$

$S_{768} = \sqrt{2-\sqrt{4-(.01636228)^2}} = .00818121 \quad \therefore P_{768} = 6.28316941$

489. Perimeters of Successive Circumscribed Polygons. By a formula similar to that of § 486, if *circumscribed* polygons of 4, 8, 16, etc., sides are used, the following results are obtained:

Number of Sides	Length of Each Side	Perimeter
4	2.000000	8.000000
8	0.828428	6.627418
16	0.397824	6.365196
32	0.196984	6.303450
64	0.098254	6.288236
128	0.049078	6.284448
256	0.024544	6.283500
512	0.012272	6.283264
1024	0.006136	6.283205
2048	0.003068	6.283190

VALUE OF PI AND AREA OF THE CIRCLE

490. Approximate Length of the Circle. From the tables in §§ 488, 489, we know that the circumference of a circle whose radius is 1 is greater than 6.283169 and less than 6.283190. Hence, the nearest approximation to four places of decimals is 6.2832.

491. The Approximate Value of π. Since in the above computations the diameter of the circle is 2, the approximate ratio of the circumference to the diameter is

$$\frac{6.2831694}{2} = 3.1415847, \text{ or } \frac{6.283190}{2} = 3.141595$$

according as the *inscribed* or *circumscribed* polygons are used. That is, these approximations of π agree to four decimal places. The nearest 5-place approximation is 3.14159.

492. The Areas of Circumscribed Polygons. By § 479 the area of each circumscribed polygon is half the product of the perimeter and the apothem, which in this case is the radius of the circle. That is, for every such polygon the area is

$$a = \tfrac{1}{2} p \cdot r,$$

where p is the perimeter of the polygon and r is the radius of the inscribed circle.

We now assume that a circle contains a ***definite area*** *which can be* ***approximated*** *as closely as we please by taking the areas of the successive circumscribed polygons.*

493. The Area of the Circle. Since the perimeters of the circumscribed polygons can be made to approximate the length c of the circle as nearly as we please, and since $c = 2\pi r$, this constitutes an informal proof of the following:

494. THEOREM VII. *The area of a circle is one half the circumference times the radius, or in symbols*

$$\text{area of circle} = \tfrac{1}{2} \cdot 2\pi r \cdot r = \pi r^2.$$

RATIO OF THE AREAS OF TWO CIRCLES

495. THEOREM VIII. *The areas of two circles are in the same ratio as the squares of their radii, or as the squares of their diameters.*

Proof. Since the area of a circle is πr^2, if we have two given circles whose radii are r and r' and whose diameters are d and d', then the ratio of their areas a and a' is

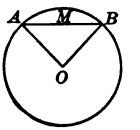
$$\frac{a}{a'} = \frac{\pi r^2}{\pi r'^2} = \frac{r^2}{r'^2} = \frac{d^2}{d'^2}.$$

496. The area of a sector bears the same ratio to the area of the circle as the angle of the sector does to the perigon.

E.g. the area of the sector whose arc is a quadrant is one fourth of the area of the circle, that is, $\frac{\pi r^2}{4}$. The area of a semicircle is $\frac{\pi r^2}{2}$.

497. The area of a segment of a circle is known if that of the sector having the same arc, and of the central triangle on the same chord, can be determined.

E.g. the area of a segment made by the chord of a quadrant is found as follows:



The area of the sector AOB is $\frac{\pi r^2}{4}$.

The area of the $\triangle AOB$ is $\frac{1}{2} r \cdot r = \frac{r^2}{2}$.

Hence the area of the segment M is $\frac{\pi r^2}{4} - \frac{r^2}{2} = \frac{r^2}{4}(\pi - 2)$.

Note. — The accompanying figure shows a circle cut into sectors by a series of radii. Each sector approximates the shape of a triangle, whose altitude is the radius of the circle, and whose base is an arc of the circle. Since the sum of the areas of these triangles is the product of the altitude and the sum of the bases, we obtain a verification of the theorem that the area of the circle is one half the product of the circumference and radius.

Archimedes (287–212 B.C) was without doubt the greatest mathematician of his time. His standing in the old world was comparable to that of Newton in the modern world.

His work is known not through any extended treatise, like that of Euclid, but through a series of monographs on practically every mathematical subject then known; including physics, mechanics, astronomy, and many phases of geometry, such as works on parallel lines, right-angled triangles, inscribed polygons, and systems of tangent circles.

Of special interest is his mensuration of the circle, in which he found the value of π to lie between $3\frac{1}{7}$ and $3\frac{10}{71}$ by the use of inscribed and circumscribed polygons of ninety-six sides.

PROBLEMS INVOLVING NUMERICAL COMPUTATION

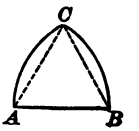
Note. — In the following problems, unless otherwise specified, use the value $\pi = 3\frac{1}{7}$, which differs from 3.14159 by about $\frac{1}{10}$ of one per cent.

1. The diameter of a circle is 8 inches. Find the circumference and area.

2. The circumference of a circle is 10 feet. Find its radius and area.

3. The area of a circle is 24 square inches. Find its radius and circumference.

4. ABC is a Gothic arch based on the equilateral triangle ABC. Find the area of the whole arch, if $AB = 3$ feet.



Suggestion. Find the area of the *sector* with center A, arc BC, and radii AB and AC, and add to this the area of the circle-segment whose chord is AC.

5. Find the ratio of the sides of squares inscribed in, and circumscribed about, the same circle. If a side of an inscribed square is 3 in., find the perimeter of a circumscribed square.

6. Find the ratio of the perimeters of regular hexagons inscribed in, and circumscribed about, the same circle. If a side of an inscribed regular hexagon is 6 in., find the perimeter of a regular circumscribed hexagon.

7. Find the ratio of the perimeters of regular triangles inscribed in, and circumscribed about, the same circle. If a side of a regular circumscribed triangle is 10 in., find the perimeter of a regular inscribed triangle.

8. Find the ratio of the areas of regular triangles inscribed in, and circumscribed about, the same circle. Also find the ratio of the areas of such squares and such hexagons.

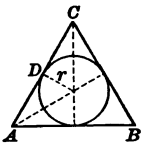
9. Find the area of a regular triangle inscribed in a circle whose radius is 6 inches; also of a regular hexagon.

10. Find the area of a regular triangle circumscribed about a circle whose radius is 6 inches; also of a regular hexagon.

PROBLEMS INVOLVING ALGEBRAIC COMPUTATION

1. Find the area of a regular triangle inscribed in a circle whose radius is r inches.

2. A regular triangle is circumscribed about a circle of radius r. Find its area.

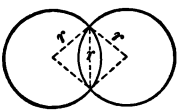


3. A regular triangle of area 36 square inches is inscribed in a circle. Find the radius of the circle.

4. One half the area of a regular hexagonal plot of ground is occupied by a walk of uniform width extending around it. Find by algebraic computation the width of the walk if a side of the plot is 12 feet.

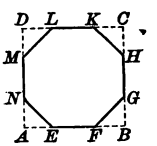
5. The difference between the areas of the squares circumscribed about two circles is 50 square inches and the difference of the diameters of the circles is 4 inches. Find each diameter.

6. Two equal circles of radius r intersect so that their common chord is equal to r. Find the area of the figure which lies within both circles.



7. Two circles of radii r and r' are tangent internally. Find the length of a chord of the larger circle tangent to the smaller if the chord (a) is parallel to the line of centers, (b) is perpendicular to the line of centers, (c) meets the larger circle at the same point as the line of centers.

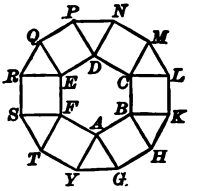
8. In the figure, $ABCD$ is a square and $EFGHKLMN$ is a regular octagon. (a) If $EF = a$, find AB. (b) If $AB = s$, find EF. (c) If $AB = s$, find the apothem. (d) If $AB = s$, find the radius of the octagon.



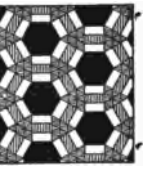
9. Find the radius of a circle if the area of its regular inscribed triangle is a.

10. Find the radius of the circle if the area of the regular circumscribed triangle is A.

PROBLEMS INVOLVING REGULAR POLYGONS



1. The figure *GHKLMP* ··· is a regular twelve-sided polygon (dodecagon). The figures *ABHG*, *BCLK*, etc., constructed on its sides are squares. Prove that a regular hexagon, *ABCDEF*, is thus formed.



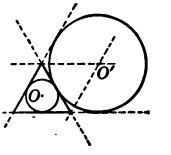
2. By means of the above figure find the area of a regular dodecagon whose sides are 6 inches. Notice that the dodecagon consists of the regular hexagon in the center, the six equilateral triangles, and the six squares.

Note how this figure enters into the accompanying tile design.

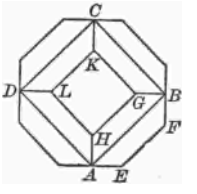
3. Find an expression for the area of a regular dodecagon whose sides are each a.

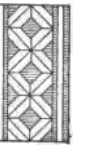
4. Find the apothem of a regular dodecagon by dividing its area by half the perimeter. Also find it by finding the apothem of the hexagon and then adding a to this.

5. Find the area of a circle inscribed in a regular dodecagon whose side is 4; one whose side is 9.



6. If the inscribed and escribed circles O and O' of an equilateral triangle are constructed as shown in the figure, find the ratio of their radii. Does this ratio depend upon the size of the triangle?





7. The middle points, *A*, *B*, *C*, *D*, of alternate sides of a regular octagon are joined as shown in the figure. *AH* is perpendicular to *AE* and equal to it. *BG*, *CK*, and *DL* are constructed in the same manner.

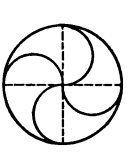
Prove that *ABCD* and *HGKL* are squares.

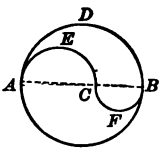
MISCELLANEOUS PROBLEMS AND THEOREMS

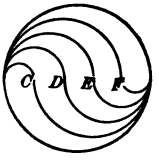
1. Chords are drawn through a fixed point on a circle. Find the locus of points which divide them in a fixed ratio.

2. Two concentric circles are such that one divides the area of the other into two equal parts. Find the ratio of the radii of the circles.

3. In a given circle two diameters are drawn at right angles to each other. On the radii thus formed as diameters semicircles are constructed. Show that the four figures thus formed are equal in all respects.







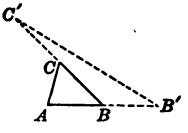
4. Let C be any point on the diameter AB of a circle.

(a) Compare the length of the arc ADB with the sum of the lengths of the arcs AEC and CFB.

(b) Show that if $AB = 3\,CB$, then the area of the figure $AECFBD$ is one third the area of the circle.

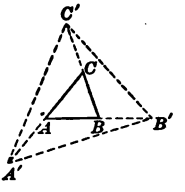
(c) Show that if $AB = m \times CB$, then the area inclosed by these arcs is one mth of the area of the circle.

5. By means of arcs constructed as shown in the third figure above, divide the circle into any given number of parts having equal areas.



6. Two sides AB and BC of a triangle are extended their own lengths to B' and C' respectively. Compare the areas of the triangles ABC and $BB'C'$.

7. The three sides of a triangle ABC are extended to A', B', C' as shown in the figure. Compare the areas of the $\triangle ABC$ and $A'B'C'$, if $BB' = AB$, $CC' = BC$, and $AA' = CA$.



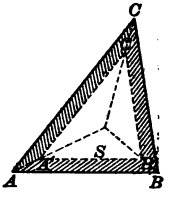
8. A square is inscribed in a circle. Find the ratio between the areas of the square and the circle.

9. A square is circumscribed about a circle. Find the ratio between their areas.

Do the ratios required in Examples 8 and 9 depend upon the radius of the circle?

10. If two lines are drawn parallel respectively to the two sides of an angle and at equal distances from these sides, they will meet, if sufficiently produced, on the bisector of the angle.

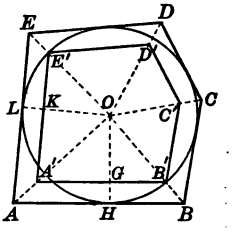
11. A triangular plot of ground *ABC* is to be laid out as a flower bed with a walk of uniform width extending around it. Prove that the flower bed is similar to the original triangle.



12. Is the construction proposed in Example 11 always possible in the case of a square? of a rhombus? Prove.

13. If any polygon is circumscribed about a circle, show that the bisectors of all its angles meet in a point.

14. Given any polygon circumscribed about a circle. Within it draw segments parallel to each of its sides and at the same distance from each side. Show that these segments form a polygon similar to the first.

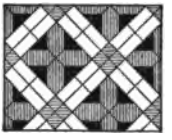


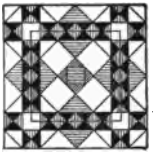
15. If the bisectors of all the angles of a polygon meet in a point prove that a circle may be inscribed in it (tangent to all its sides).

16. The following is an ancient Chinese problem. "In the center of a circular pool of water 10 feet in diameter there was a reed extending one foot above the water. When the reed was bent over so that it just reached the top of the water it also just reached the edge of the pool. How deep was the water?".

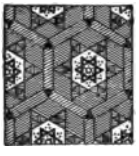
PRACTICAL DESIGNS BASED ON REGULAR POLYGONS

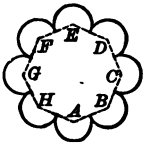
Designs Based on Squares

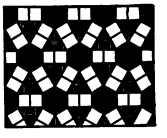


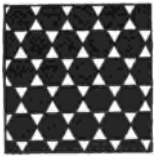


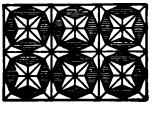
Designs Based on Regular Hexagons and Octagons

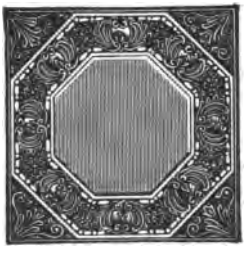


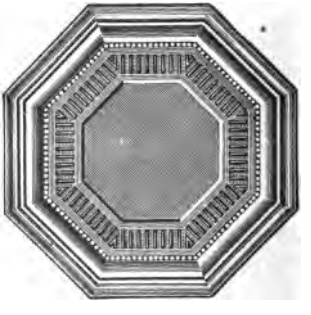






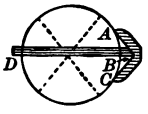




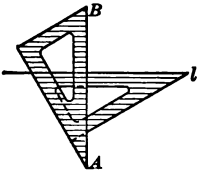


PRACTICAL APPLICATIONS

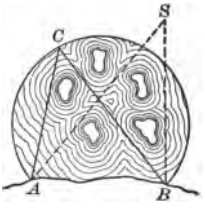
1. The figure shows an instrument used for finding the center of a circular disk. $AB = BC$ and DB bisects $\angle ABC$. Show that the line DB passes through the center of the circle, and that if two lines are drawn across the disk by means of this instrument the lines will meet in the center.



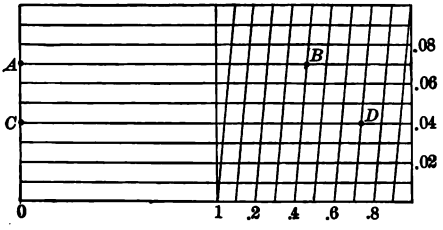
2. Show that by placing a triangular ruler (having one right angle) in two positions, as shown in the figure, the line AB is made perpendicular to the line l.



3. On a dangerous coast there are lighthouses at points marked A and B. It is known that the dangerous rocks are within a certain segment of a circle passing through A and B and containing a certain angle ACB. If the ship S keeps at a distance making the angle ASB less than the known angle ACB, the ship is safe. Explain. The angle ACB is called the danger angle.

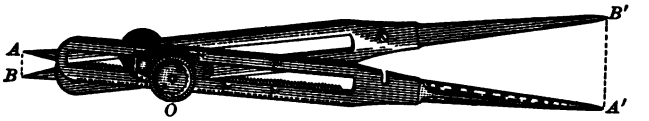


4. The so-called diagonal scale is used to measure distances very accurately. In the figure, what are the distances AB and CD? Explain fully.



5. In estimating standing timber it is often necessary to find the height of a tree up to a certain point on it. Explain how this can be done by measuring the shadow cast by the tree up to this point and then finding the length of a shadow cast by a staff of known length.

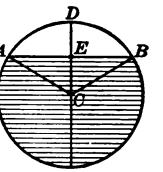
6. An instrument called proportional dividers is used to enlarge or reduce maps. The screw at O may be set so as to



make the ratio $OA : OA'$ equal to the scale on which it is desired to copy the map. Show that $AB : A'B' = OA : OA'$.

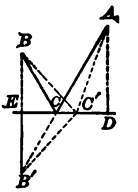
7. Two water pipes, 8 and 10 inches respectively in diameter, are to be supplied by a single pipe. What must be its diameter?

8. A water pipe 12 inches in diameter is filled to a depth of 9 inches. Find the area of the cross section of the water.



9. Given a circle and a carpenter's square. Show how to locate the center of the circle.

10. Given a straight line and two points A and B on the same side of it. Find a point C on the line such that the sum of the segments AC and BC shall be the least possible.

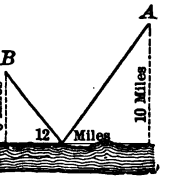


Suggestion. In the figure, let BB' be $\perp ED$ and $EB' = EB$. Draw AB' meeting the line in C. Then C is the required point. For let C' be any other point on the line. Then $AC' + C'B > AC + CB$. The proof depends upon § 191 and Ax. 16, § 71. Give it in full detail.

11. If, in the figure preceding, AD is perpendicular to the line, prove that $\triangle ADC \sim \triangle CBE$, and hence show that $AD : BE = DC : CE$.

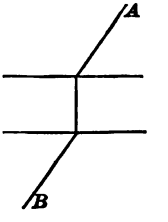
Note. — Since $\angle ECB = \angle DCA$ and it is known that a ray of light passing from A to B after being reflected also makes these angles equal, therefore the ray passes over the shortest possible path.

12. Two towns, A and B, are 10 and 6 miles respectively from a river, and A is 12 miles farther up the river than B. A pumping station is to be built which shall serve both towns. Where must it be located so that the total length of water main to the two towns shall be the least possible?

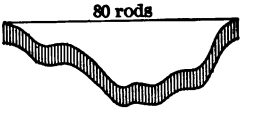


13. Two factories are situated on the same side of a railway at different distances from it. A spur is to be built to each factory and these are to join the railway at the same point. State just what measurements must be made and how to locate the point where these spurs should join the main line in order to permit the shortest length of road to be built.

14. Two points A and B, on opposite sides of a straight stream of uniform width, are to be connected by a road and a bridge crossing the stream at right angles. Find by construction the location of the bridge so as to make the total path from A to B the shortest possible.



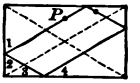
15. A solid board fence 5 feet in vertical height running due north and south is to be built across a valley, connecting two points of the same elevation. Find the number of square feet in the fence if the horizontal distance is 80 rods.



16. Are the data given in the preceding problem sufficient to solve it if the fence required is to be an ordinary four-board fence, each board 6 inches wide? Explain fully the difference between these conditions.

17. A circular pond is surrounded by a gravel walk, such that the area of the walk is equal to the area of the pond. What is the ratio of the radius of the pond to the width of the walk?

18. A billiard ball is placed at a point P on a billiard table. In what direction must it be shot in order to return to the same point after hitting all four sides?

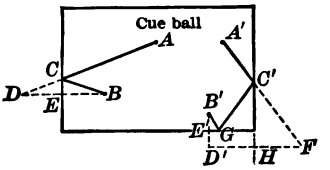


The angle at which the ball is reflected from a side is equal to the angle at which it meets the side, that is, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.

Suggestion. (*a*) Show that the opposite sides of the quadrilateral along which the ball travels are parallel.

(*b*) If the ball is started parallel to a diagonal of the table, show that it will return to the starting point.

19. Find the direction in which a billiard ball must be shot from a given point on the table so as to strike another ball at a given point after first striking one side of the table.



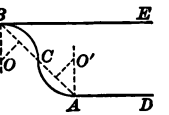
Suggestion. Construct $BE \perp$ to that side of the table which the ball is to strike and make $ED = BE$.

20. The same as the preceding problem except that the cue ball is to strike two sides of the table before striking the other ball.

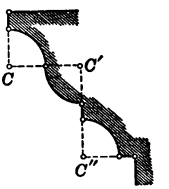
Suggestion. $B'E' = E'D'$, $D'H = HF$.

21. Solve Example 20, if the cue ball is to strike three sides before striking the other ball, — also if it is to strike all four sides.

22. Given two parallel lines BE and AD, to construct arcs which shall be tangent to each other and one of which shall be tangent to BE at B and the other tangent to AD at A.

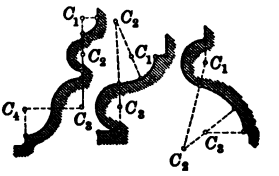


Solution. Draw AB and bisect this segment at C; construct $\perp$ bisectors of AC and BC. From A and B draw $\perp$s to AD and BE respectively, thus locating the points O and O'.



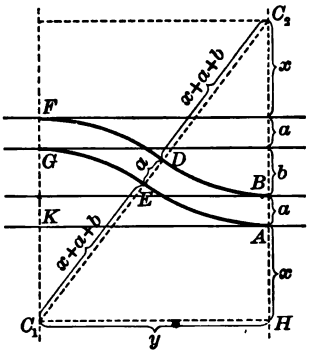
Prove that O and O' are the centers of the required arcs. Show that O, C, and O' lie in a straight line and use the theorem of § 262.

The construction in Example 22 occurs in architectural designs and in many other applications. In the accompanying designs pick out all the arcs that are tangent to each other and also the points of tangency.



SCROLL WORK

23. A cross-over track is constructed as shown in the figure. The rails of the curved track are tangent to those of the main line at A, B, F and G. The curves are tangent to each other at D and E. The arcs GE and BD have equal radii, as have the arcs AE and DF. C_1 is the center of the arcs GE and FD and C_2 is the center of the arcs BD and AE.



(a) Prove that C_2HC_1 is a right triangle.

(b) If the distance between the tracks is b feet and the distance between the rails in each track (the gauge) is a feet, show that

$$(2x + 3a + 2b)^2 = y^2 + (2x + 2a + b)^2.$$

Since a and b are given, this equation may be solved for x in terms of y or for y in terms of x.

Hence if the distance AK is known, we may use this equation to compute the radii of the arcs used in constructing the figure.

On the other hand, if the radii of the arcs are known, we may compute the distance AK.

This is a very common problem in railway construction. The construction is also used in laying out a curved street to connect two parallel streets.

24. If in the preceding problem $a = 4$ feet $8\frac{3}{4}$ inches, $b = 9$ feet, and $AK = 200$ feet (an actual case), find the radii of the arcs.

APPENDIX TO PLANE GEOMETRY

In addition to the supplementary matter found at various points in the preceding pages, three topics of special interest are treated in the following pages.

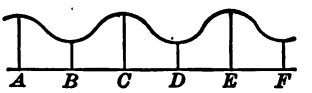
I. MAXIMA AND MINIMA

498. Definitions. Of all geometric figures fulfilling certain conditions it often happens that some one is greater than any other, in which case it is called a **maximum.** Or it may happen that some one is less than any other, in which case it is called a **minimum.**

E.g. of all chords of a circle the diameter is the *maximum*, and of all segments drawn to a line from a point outside it the perpendicular is the *minimum*.

In the following theorems and exercises the terms *maximum* and *minimum* are used as above defined. However, a geometric figure is often thought of as *continuously varying in size*, in which case it is said to have a **maximum** at *any* position where it may *cease to increase and begin to decrease*, whether or not this is the greatest of all its *possible* values. Likewise it is said to have a **minimum** at any position where it may *cease to decrease and begin to increase.*

E.g. if in the figure a perpendicular from a point in the curve to the straight line be moved continuously parallel to itself, the length of this perpendicular will have maxima at A, C, and E, and minima at B, D, and F.



Certain simple cases of maxima and minima problems already considered will be recalled in the following exercises.

EXERCISES

1. Show that of all chords through a given point within a circle, not the center, the diameter is a maximum and the chord perpendicular to the diameter is a minimum.

2. Of all line-segments drawn to meet the circle from a point within that is a maximum which passes through the center, and a minimum which, if produced in the opposite direction, would pass through the center.

3. Of all line-segments drawn to meet the circle from a point outside that is a maximum which meets it after passing through the center, and that is a minimum which, if produced, would pass through the center.

4. Show that if a square and a rectangle have equal perimeters, the square has the greater area.

Suggestion. If s is the side of the square and a and b are the altitude and base of the rectangle respectively, then $2b + 2a = 4s$.

Hence, $s^2 = \frac{b^2 + a^2 + 2ab}{4} = \frac{(b^2 - 2ab + b^2) + 4ab}{4} = \frac{(b-a)^2}{4} + ab.$

5. Use the preceding exercise to find that point in a given line-segment which divides it into two such parts that their product is a maximum.

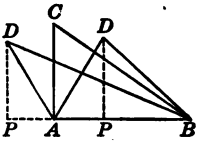
6. Find the point in a given line-segment such that the sum of the squares on the two parts into which it divides the segment is a minimum.

Suggestion. If a and b are the two parts and k the length of the segment, then $a + a = k$ and $a^2 + 2ab + b^2 = k^2$,

or, $a^2 + b^2 = k^2 - 2ab$.

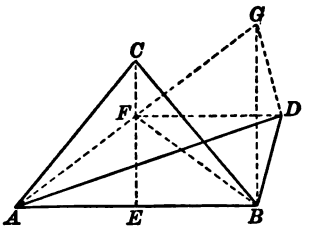
Hence, $a^2 + b^2$ is least when $2ab$ is greatest. Now apply Ex. 5.

7. Of all triangles having two given sides, that in which these sides include a right angle has the maximum area.



Suggestion. Show that the *altitude* is greatest when the angle is a right angle.

499. THEOREM. *Of all triangles having equal perimeters and the same base, the isosceles triangle has the maximum area.*



Given $\triangle ABC$ **isosceles and having the same perimeter as** $\triangle ABD$.

To prove that area $\triangle ABC >$ area $\triangle ABD$.

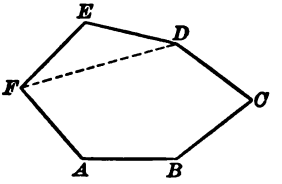
Outline of proof. Draw $CE \perp AB$. Construct $\triangle AFB$ having its altitude FE the same as that of $\triangle ABD$. Prolong AF making $FG = AF$. Draw GB and GD. The object is to prove that EF, the altitude of $\triangle ABD$, is less than EC, the altitude of $\triangle ABC$.

(1) Show that $\triangle AFB$, FBG, and GBD are all isosceles, for which purpose it must be shown that $GB \perp AB$ and $FD \parallel AB$.

Then $AD + DB = AD + DG > AG$.
Or $AF + FB < AD + DB$.
But $AD + DB = AC + CB$.
Hence, $AF + FB < AC + CB$.
(2) Show that $AF < AC$,
and hence that $EF < EC$.
(3) Use the last step to show that
area $\triangle ABC >$ area $\triangle ABD$.
Give all the steps and reasons in full.

500. COROLLARY. *Of all triangles having equal areas and standing on the same base, that which is isosceles has the least perimeter.*

501. THEOREM. *Of all polygons having equal perimeters and the same number of sides, the one with maximum area is equilateral.*

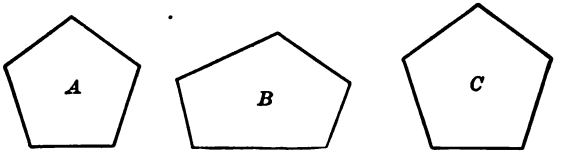


Given *ABCDEF* the maximum of polygons having equal perimeters and the same number of sides.

To prove that $AB = BC = CD = DE = EF = FA$.

Suggestion. Show that every $\triangle$, such as FDE, is isosceles by use of the preceding theorem.

502. THEOREM. *Of all polygons having equal areas and the same number of sides, the regular polygon has the minimum perimeter.*



Given polygons *A* and *B* with the same number of sides and equal areas, *A* being regular and *B* not.

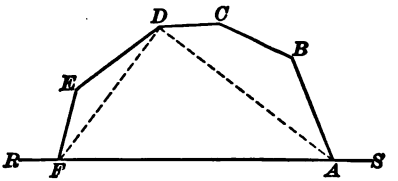
To prove that the perimeter of A is less than that of B.

Outline of Proof. Construct a regular polygon C having the same number of sides and same perimeter as B.

Then, area of $B <$ area of C, *i.e.*, area of $A <$ area of C.

Hence the perimeter of A is less than that of C, since A and C are similar. (Use § 474 and then § 420.) That is, perimeter of $A <$ perimeter of B.

503. THEOREM. *The polygon with maximum area which can be formed by a series of line-segments of given lengths, starting and ending on a given line, is that one whose vertices all lie on a semicircle constructed on the intercepted part of the given line as a diameter.*



Given the line-segments *AB*, *BC*, *CD*, *DE*, and *EF* meeting the line *RS* at *A* and *F* so as to form the polygon *ABCDEF* with maximum area.

To prove that *B*, *C*, *D*, and *E* lie on the semicircle whose diameter is *AF*.

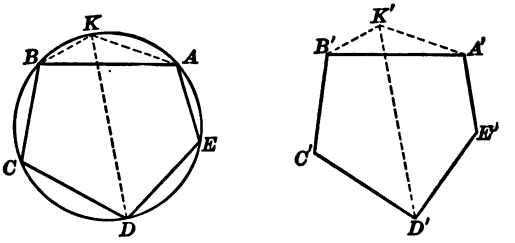
Proof: Suppose any vertex as *D* does not lie on this semicircle. Join *D* to *F* and to *A*. Then $\angle ADF$ is not a right angle, else it would be inscribable in a semicircle (§ 285).

If now the extremities *A* and *F* be moved in or out on the line *RS*, keeping *AD* and *DF* unchanged in length, and keeping the figures *FDE* and *ABCD* unchanged in shape and size, till $\angle ADF$ becomes a right angle, then $\triangle ADF$ will be *increased* in area (Example 7, page 285), while the rest of the polygon is unchanged in area.

This would increase the total area of the polygon *ABCDEF*, which is contrary to the hypothesis that this is the polygon with maximum area. Hence the vertex *D* must lie on the semicircle.

In the same manner it can be proved that each vertex lies on the semicircle.

504. THEOREM. *Of all polygons with the same number of sides equal in pairs and taken in the same order, the one with maximum area is that one which can be inscribed in a circle.*



Given a polygon *ABCDE* inscribed in a circle and *A'B'C'D'E'* not inscribable but with sides equal to those of *ABCDE*.

To prove that area $ABCDE >$ area $A'B'C'D'E'$.

Proof: Draw the diameter DK and draw AK and BK.

On $A'B'$ construct $\triangle A'B'K' = \triangle ABK$ and draw $D'K'$.

The circle whose diameter is $D'K'$ cannot pass through all the points A', B', C', and E', according to the hypothesis.

If either A' or E' is not on this circle, then

$$\text{area } AKDE > \text{area } A'K'D'E'. \quad \text{(Why?)} \quad (1)$$

Likewise if either B' or C' is not on the circle, then

$$\text{area } KBCD > \text{area } K'B'C'D'. \quad \text{(Why?)} \quad (2)$$

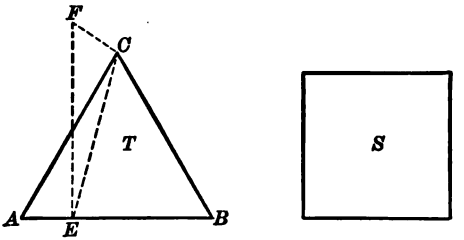
In any case, area $AKBCDE >$ area $A'K'B'C'D'E'$. (3)

By construction, $\triangle AKB = \triangle A'K'B'$. (4)

From (3) and 4, area $ABCDE >$ area $A'B'C'D'E'$.

505. COROLLARY. *Of all polygons having equal perimeters and the same number of sides, the regular polygon has the maximum area.*

506. THEOREM. *Of all regular polygons having equal perimeters, that which has the greatest number of sides has the maximum area.*



Given T **a regular triangle and** S **a square having the same perimeter.**

To prove that area $S >$ *area* T.

Proof: Take any point E in AB. Draw CE, and on CE construct $\triangle CEF = \triangle AEC$.

Then polygon $BCFE$ has a perimeter equal to that of T and the same area, and has the same number of sides as S.

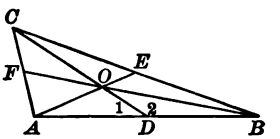
Hence, area $S >$ area $BCFE$. § 505

That is, area $S >$ area T.

In like manner it can be shown that a regular polygon of five sides has a greater area than a square of equal perimeter, and so on for any number of sides.

EXERCISE

Of the three medians of a scalene triangle that one is the shortest which is drawn to the longest side.



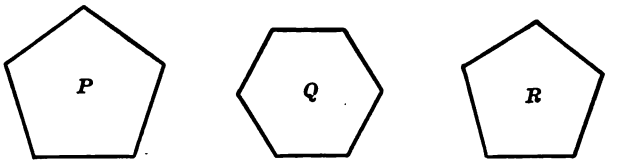
Suggestion. If $AC < BC$, to show that $BF > AE$.

In $\triangle ADC$ and BDC, $AD = DB$ and DC is common.

But $AC < BC$. Hence $\angle 1 < \angle 2$ (Why ?).

Now use $\triangle ADO$ and BDO to show that $BO > AO$ and hence that $BF > AE$.

507. THEOREM. *Of all regular polygons having a given area, that one which has the greatest number of sides has the least perimeter.*



Given regular polygons P and Q such that area P = area Q while Q has the greater number of sides.

To prove that perimeter of Q < perimeter of P.

Proof: Construct a regular polygon R having the same number of sides as P and the same perimeter as Q.

Then	area $R <$ area Q.	§ 506
But	area $P =$ area Q.	(*By hypothesis*)
Hence,	area $R <$ area P.	

Therefore, perimeter of $R <$ perimeter of P. §§ 420, 474.
But R was constructed with a perimeter equal to that of Q.
Hence perimeter of $Q <$ perimeter of P.

EXERCISES

1. Of all polygons of a given number of sides inscribed in a given circle, that one which is regular has the maximum area and the maximum perimeter.

2. Of all polygons of a given number of sides circumscribed about a given circle, that one which is regular has the least perimeter and the least area.

3. Solve examples 10–14, pages 280, 281, as practical applications of this subject.

II. VARIABLES AND LIMITS

508. Variables and Functions. It is often useful to think of a geometric figure as **continuously varying** in size and shape.

E.g. if a rectangle has a fixed base, say 10 inches long, but an altitude which varies continuously from 3 inches to 5 inches, then the area varies continuously from $3 \cdot 10 = 30$ to $5 \cdot 10 = 50$ square inches.

We may even think of the altitude as starting at zero inches and increasing continuously, in which case the area starts at zero and increases continuously.

The altitude which we think of as varying at our pleasure is called the **independent variable**, while the area, being dependent upon the altitude, is called the **dependent variable.**

The dependent variable is sometimes called a **function** of the independent variable, meaning that the two are connected by a definite relation such that, for any definite value of the independent variable, the dependent variable also has a definite value.

Thus, in the formula for the area of a rectangle, $a = bh$, if b is fixed and h varies, then a is a function of h and for every value of h there is determined a definite value of a.

509. The Notion of a Limit. If a regular polygon is inscribed in a circle of fixed radius, and if the number of sides of the polygon be continually increased, for instance by repeatedly doubling the number, then the apothem, perimeter, and area are all variables depending upon the number of sides. That is, each of these is a function of the number of sides.

Now the greater the number of sides the more nearly does the apothem equal the radius in length. Indeed, it is evident that the difference between the apothem and the radius will ultimately become less than any fixed number, however small. Hence we say that the apothem *approaches the radius as a limit* as the number of sides increases indefinitely.

Similarly the perimeters of the polygons may be made as nearly equal to the circumference as we please by making the number of sides sufficiently great.

Hence we may define the circumference of a circle as the *limit of the perimeter* of a regular inscribed polygon as the number of sides increases indefinitely.

The circumference of a circle may also be defined as the limit of the perimeter of a circumscribed polygon as the number of sides is increased indefinitely.

That these two definitions of the circumference are consistent may be inferred from §§ 486–491.

Likewise we may define the area of a circle as the *limit of the area* of the inscribed or the circumscribed polygon as the number of sides is increased indefinitely.

The notion of a limit may be used to define the length of a line-segment which is incommensurable with a given unit segment.

Thus, the diagonal d of a square whose side is unity is $d = \sqrt{2}$. Hence d may be defined as the *limit of the variable line-segment* whose successive lengths are 1, 1.4, 1.41, 1.414, ···. See § 298.

In like manner, the length of any line-segment, whether commensurable or incommensurable with the unit segment, may be defined in terms of a limit.

Thus, if a variable segment is increased by successively adding to it one half the length previously added, then the segment will approach a limit. If the initial length is 1, then the successive additions are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., and the successive lengths are 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$, $1\frac{31}{32}$, etc. Evidently this segment approaches the limit 2.

Hence 2 may be defined as the limit of the variable segment, whose successive lengths are 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, etc., as the number of successive additions is increased indefinitely.

The functional relation between variables and the idea of a limit as illustrated above are two of the most important concepts in all mathematics.

510. Constant. A quantity which remains fixed in value throughout a discussion is called a *constant.*

E.g. the base of the rectangle mentioned in § 508.

511. Variable. A quantity which continuously changes in value, or which takes on a succession of different values, is called a *variable.*

E.g. the altitude and the area mentioned in § 508.

512. Limit of a Variable. If a variable may be made to approach a certain constant quantity in such a way that the difference between the constant and the variable becomes and remains less than any assignable value, however small, then the constant is called the *limit of the variable.*

E.g. the fixed circumference of the circle is the limit of the variable perimeters of the polygons mentioned in § 509.

SIGHT WORK

1. Find the limit of a variable line-segment whose initial length is 6 inches, and which varies by successive additions each equal to one half the preceding, the first addition being 2 inches.

2. Construct a right triangle whose sides are 1 and 2. By approximating a square root, find five successive lengths of a segment which approaches the length of the hypotenuse as a limit.

3. If one tangent to a circle is fixed and another is made to move so that their intersection point approaches the circle, what is the limiting position of the moving tangent? What is the limit of the measure of the angle formed by the tangents?

4. The arc AB of 74° is the greater of the two arcs intercepted between two secants meeting at C outside the circle. The points A and B remain fixed while C moves up to the circle. What angle is the limit of the variable angle formed by the varying secants? The limit of the measure of this angle?

5. If in the preceding the secants meet within the circle, what is the limit of their angle and also of the measure of this angle as the intersection point moves up to the center of the circle?

III. THEORY OF LIMITS

513. The Incommensurable Cases. We have seen, in § 298, that there are segments which are *incommensurable;* that is, which have no common unit of measure, — for instance, the side and the diagonal of a square.

For *practical purposes* the lengths of such segments are approximated to any desired degree of accuracy, and their ratios are understood to be the ratios of these approximate numerical measures. See §§ 300–304.

But for *theoretical purposes* it is important to consider these incommensurable cases further; just as in algebra we not only *approximate* such roots as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., but we also deal with these surds as *exact* numbers.

Instances of this kind occur in such operations as

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=3-2=1.$$

While the length of the diagonal of a unit square cannot be expressed as an integer or as a rational fraction, that is, as the quotient of two integers, we nevertheless think of such a segment as having a *definite length*, or what is the same thing, a *definite ratio* with the unit segment forming the side of the square.

We now fix our attention on the *incommensurable ratios themselves*, and the method of determining them, rather than on the *process of approach* and the practical computation based on it.

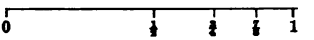
514. Representation by Irrational Numbers. In general, the ratio between any two incommensurable geometric magnitudes of the same kind may be represented by what is called an *irrational number;* that is, a number which is *neither an integer nor a quotient of two integers.*

Examples of irrational numbers are the surds, such as $\sqrt{2}$, $\sqrt[3]{5}$, etc., and the number π. See § 491.

515. Numbers Defined by Sequences. The following is a method for determining any number, whether rational or irrational. For simplicity it is applied first to the integer 1.

Throughout this discussion the words "point on a line" and "number" will be used interchangeably.

On a straight line mark a certain point 0 (zero), and one unit to the right of it mark another point 1. Also lay off points such that their distances from 0 are, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, ...



If this sequence of points is carried ever so far, it will never reach the point 1. If, however, we select a point to the left of 1, no matter how near, we may always go far enough along this sequence to reach points between it and 1. That is, 1 is the limit of the terms of this sequence. See § 512.

The point 1 has two definite relations to this sequence:

(*a*) *Every point of the sequence is to the left of* 1.

(*b*) *For any fixed point to the left of* 1 *there are points of the sequence between it and* 1.

We see that 1 is the *only* point on the whole line such that both (*a*) and (*b*) are true of it. For every point to the right of 1 (*a*) is true, but (*b*) is not. For every point to the left of 1 (*b*) is true, but (*a*) is not.

It follows therefore that, while the points of the sequence merely *approach* the point 1 as a limit, the sequence, *taken as a whole, serves to distinguish that point from all other points on the line.*

In the above sequence, the terms continually *increase.* The number 1 may be determined equally well by a *decreasing* sequence. For instance, the sequence $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, $1\frac{1}{16}$, ··· has the following properties which distinguish the point 1 from all other points on the line:

(*a*) *Every point of the sequence is to the right of* 1.

(*b*) *For any fixed point to the right of* 1 *there are points of the sequence between it and* 1.

516. **Infinite Sequences.** An endless sequence of either sort just described is called an **infinite sequence.**

Not every infinite sequence serves to single out a definite point in the manner shown above. Thus the sequence 1, 2, 3, 4, ··· fails to do so, because its terms grow large beyond all bound. Such sequences are said to be *unbounded*, while the sequences $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, ··· and $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, ··· are *bounded.*

Again, the sequence 1, 2, 1, 2, 1, 2, ··· fails to single out a definite point. This sequence is said to be *oscillating*, since its terms increase, then decrease, then increase, etc., while $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, ··· and $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, ··· are *non-oscillating.*

517. **Limit of a Sequence.** The number 1 is said to be the *least upper bound* of the sequence $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, ···. That is, 1 is the *smallest number* beyond which the sequence does not go. 1 is also said to be the *limit of the sequence.* See § 512.

Similarly, 1 is the *greatest lower bound* or the *limit* of the sequence $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, ···; that is, the *greatest number* such that the sequence contains no number less than it.

In the manner described above any integer may be determined by an increasing or a decreasing infinite sequence.

E.g. the number 2 is the limit of the increasing sequence 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$, ···, or of the decreasing sequence $2\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{1}{8}$, $2\frac{1}{16}$, ····

Two different sequences, both increasing or both decreasing, may also define the same number.

E.g. Each of the increasing sequences $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, ··· and $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, $\frac{8}{9}$, ··· determines the number 1.

We notice, however, that no matter what definite number we select in either of these sequences, there is a number in the other greater than it; that is, neither sequence contains a number greater than every number in the other. See § 519.

It is apparent that any number may be defined as the limit of a sequence, and we shall assume the converse, namely, that any bounded sequence has a limit. See § 518.

518. AXIOM XXII. *Every bounded increasing sequence has a least upper bound, and every bounded decreasing sequence has a greatest lower bound.*

This axiom may also be stated:

Every bounded increasing or decreasing sequence has a limit.

This axiom simply means that every such sequence singles out a definite number, rational or irrational, in the manner stated.

Thus, if we attempt to approximate the square root of 2, we obtain a sequence 1, 1.4, 1.41, 1.414, 1.4142, ···, having for its limit a *definite number* represented by $\sqrt{2}$, which corresponds to the length of the diagonal of a square whose side is unity.

Again, if we attempt to approximate the value of π as in §§ 486–491 we obtain a sequence 3, 3.1, 3.14, 3.141, 3.1415, ···, having for its limit the definite number represented by π, which is the ratio of the circumference to the diameter of a circle.

Historical Note. — The rigorous treatment of limits has been developed during the last fifty years. The foundation on which it rests is the theory of irrational numbers, which itself was placed on a firm basis in the early seventies.

THREE FUNDAMENTAL PRINCIPLES OF LIMITS

519. THEOREM I. *Two bounded sequences determine the same number if they are equal term by term.*

Proof: If the sequences are $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$, and if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3, \cdots$, then they are one and the same sequence and hence determine the same number by Axiom XXII.

This theorem is used in the applications of limits to geometry which involve the incommensurable ratios. See §§ 526, 527.

520. THEOREM II. *Two increasing bounded sequences have the same limit if neither sequence contains a number greater than every number of the other.*

Let $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ denote two infinite sequences with limits A and B, such that there is no a greater than every b, and no b greater than every a.

To prove that $A = B$.

Proof: Suppose that A is not equal to B but is less than B. Then there must be numbers of the sequence $b_1, b_2, b_3, \cdots$ greater than A, since by § 515 there are numbers of $b_1, b_2, b_3, \cdots$ greater than any fixed number whatever which is less than B. This contradicts the hypothesis of the theorem, and hence A cannot be less than B.

In like manner we show that B is not less than A. Hence $A = B$.

521. THEOREM III. *Two decreasing bounded sequences have the same limit if neither sequence contains a number less than every number of the other.*

Proof: The proof is exactly similar to that for Theorem II.

EXERCISES

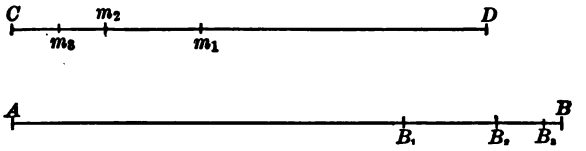
1. Show that the sequences $\frac{1}{2}, \frac{3}{4}, \frac{7}{8} \cdots$ and $\frac{2}{3}, \frac{4}{5}, \frac{6}{7} \cdots$ have the same limits.

2. Show that the sequences $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \cdots$ and $\frac{1}{3}, \frac{1}{5}, \frac{1}{7} \cdots$ have the same limits.

The applications of the theory of limits to geometry consist chiefly in showing that two numbers are equal because they are the limits of the same sequence, or of sequences having the property stated in the theorem of § 520 or of § 521.

APPLICATION OF LIMITS TO PLANE GEOMETRY

522. PROBLEM. *On a given segment AB to lay off a sequence of points B_1, B_2, B_3, of which B is the limit, such that each of the segments AB_1, AB_2, AB_3, ··· is commensurable with a given segment CD.*



Solution. Using m_1, an exact divisor of CD, as a unit of measure, lay off on AB a segment AB_1, such that the remainder B_1B is less than m_1. Then CD and AB_1 are commensurable.

Using a unit m_2, likewise a divisor of CD, and less than B_1B, lay off AB_2 such that B_2B is less than m_2. Then $AB_2 > AB_1$, and CD and AB_2 are commensurable.

Continuing in this manner, using as units of measure segments m_3, m_4, ··· each an exact divisor of CD and each less than B_2B, B_3B, ··· respectively, we obtain a sequence of segments AB_1, AB_2, AB_3, ···, each greater than the preceding and each commensurable with CD.

If the units m_1, m_2, m_3, ··· are so selected that they approach zero as a limit, it follows that B is the limit of the sequence B_1, B_2, B_3, ···.

If a different sequence of divisors of CD, as m_1', m_2', m_3', ···, is used, we obtain a sequence B_1', B_2', B_3', ···, likewise satisfying the conditions of the problem.

Any two such sequences B_1, B_2, B_3, ··· and B_1', B_2', B_3', ··· determined as above, are both *increasing* and each is such that *no* point of it is to the right of *every* point of the other.

Hence, by § 520 any two such sequences have the same limit B.

523. Limit of a Sequence of Segments. If B_1, B_2, B_3, ··· is a sequence of points on the segment AB having the limit B, then the segment AB is said to *be the limit of the sequence of segments* AB_1, AB_2, AB_3, ···.

524. Ratio of two Incommensurable Segments. The ratio of two incommensurable segments has not been explicitly defined, but it is now possible to do so as follows in terms of the *limit of a sequence.*

Consider two incommensurable segments AB and CD. Let $a_1, a_2, a_3, \cdots$ be the lengths of the segments each commensurable with CD, forming a sequence whose limit is the segment AB, and let b be the length of the segment CD.

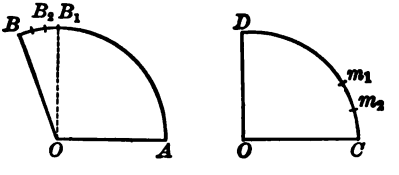
Then $\frac{a_1}{b}, \frac{a_2}{b}, \frac{a_3}{b}, \cdots$ is an increasing bounded sequence having a limit which we call R.

If $a_1', a_2', a_3', \cdots$ are the lengths of any other sequence of segments whose limit is AB, the sequence $\frac{a_1'}{b}, \frac{a_2'}{b}, \frac{a_3'}{b}, \cdots$ is another increasing bounded sequence with limit R'.

By § 520 we now know that $R = R'$.

This number R is defined as *the ratio of the incommensurable segments* AB and CD.

525. Ratio of two Incommensurable Arcs. The considerations of §§ 522, 523 apply equally well to two arcs of the same circle or of equal circles.

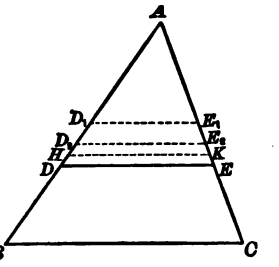


Thus, in the figure, the ratio of the two incommensurable arcs AB and CD is the limit of the sequence $\frac{AB_1}{CD}, \frac{AB_2}{CD}, \frac{AB_3}{CD}, \cdots$, and the ratio of the incommensurable angles AOB and COD is the limit of the sequence $\frac{\angle AOB_1}{\angle COD}, \frac{\angle AOB_2}{\angle COD}, \frac{\angle AOB_3}{\angle COD}, \cdots$.

526. THEOREM. *A line parallel to the base of a triangle, and meeting the other two sides, divides them in the same ratio.*

Given the $\triangle ABC$ with $DE \parallel BC$ and cutting AB and AC.

To prove that $\frac{AD}{AB} = \frac{AE}{AC}$.



Proof: Consider the case when AD and AB are incommensurable.

Let D_1, D_2, D_3, ··· be a sequence on AB whose limit is D. Through these points draw parallels to BC, meeting AC in E_1, E_2, E_3, ···.

Then E is the limit of the sequence E_1, E_2, E_3, ···.

For suppose it is not, and that there is a point K on AE such that there is no point of E_1, E_2, E_3, ··· between K and E. Then draw a line parallel to BC through K, meeting AB in H.

But there are points of the sequence D_1, D_2, D_3, *between* H and D, and hence there must be points of the sequence E_1, E_2, E_3, ··· *between* K and E, which shows that E is the limit of the sequence E_1, E_2, E_3, ···.

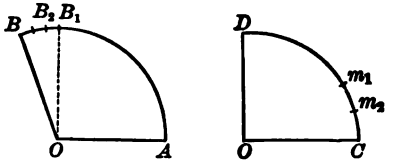
Then the sequence (1) $\frac{AD_1}{AB}, \frac{AD_2}{AB}, \frac{AD_3}{AB}$, ··· has a limit R which, by definition, is the ratio of the segments AD and AB. § 524

Similarly, the sequence (2) $\frac{AE_1}{AC}, \frac{AE_2}{AC}, \frac{AE_3}{AC}$, ··· has a limit R' which is the ratio of AE to AC.

Now $\frac{AD_1}{AB} = \frac{AE_1}{AC}, \frac{AD_2}{AB} = \frac{AE_2}{AC}, \frac{AD_3}{AB} = \frac{AE_3}{AC}$, ···. § 321

Hence, the two sequences, (1) and (2) are identical and have the same limit (§ 519). Hence $R = R'$.

527. THEOREM. *In the same circle or in equal circles the ratio of two central angles is the same as the ratio of their intercepted arcs.*



Proof: In case the arcs are commensurable the proof is obvious. For, in that case, the common measure of the arcs divides each into a number of equal parts, say m and n respectively, and these arcs subtend the same number of equal angles at the center (§ 244). Hence the ratio of the arcs is $m : n$ and the ratio of the angles is also $m : n$.

If the arcs AB and CD are not commensurable, let AB_1, AB_2, AB_3, ··· be a sequence of arcs whose limit is AB, each arc being commensurable with the arc CD.

Then the sequence $\frac{AB_1}{CD}, \frac{AB_2}{CD}, \frac{AB_3}{CD}, \cdots$ has a limit $R = \frac{AB}{CD}$.

Similarly the sequence $\frac{\angle AOB_1}{\angle COD}, \frac{\angle AOB_2}{\angle COD}, \frac{\angle AOB_3}{\angle COD}, \cdots$ has a limit $R' = \frac{\angle AOB}{\angle COD}$.

But $$\frac{AB_1}{CD} = \frac{\angle AOB_1}{\angle COD}, \frac{AB_2}{CD} = \frac{\angle AOB_2}{\angle COD}, \cdots,$$

as shown above for the commensurable case.

Hence, these two sequences are identical and have the same limit. Therefore it follows that $R = R'$.

Note. — The student should note the successive steps in the proofs of §§ 526, 527 : (*a*) Definitions of the incommensurable ratios by means of infinite sequences ; (*b*) Proof that these sequences are identical ; (*c*) Conclusion that the limiting ratios are equal.

528. **Products of Irrational Numbers.** If $a_1, a_2, a_3, \cdots$ is a sequence with limit a, then $ka_1, ka_2, ka_3, \cdots$ is a sequence whose limit is *defined* as ka where k is any number.

If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are two sequences with limits a and b respectively, then $a_1b_1, a_2b_2, a_3b_3, \cdots$ is a sequence whose limit is *defined* as ab.

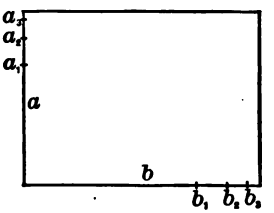
Similarly if $a_1, a_2, a_3, \cdots, b_1, b_2, b_3, \cdots$ and $c_1, c_2, c_3, \cdots$ are sequences with limits a, b, c, then abc is defined as the limit of the sequence $a_1b_1c_1, a_2b_2c_2, a_3b_3c_3, \cdots$.

For a complete treatment it would be necessary to show that these definitions of multiplication of irrational numbers are consistent with the rest of arithmetic and also that these new sequences are such as to determine definite limits. This is possible and, indeed, has been done, but it is beyond the scope of this book.

529. **Area of a Rectangle.** If the sides of a rectangle are incommensurable with the unit segment, we *define its area* as follows:

Let $a_1, a_2, a_3, \cdots$ be a sequence of rational numbers whose limit is the altitude a; and let $b_1, b_2, b_3, \cdots$ be a sequence whose limit is the base b.

Then the *area of the rectangle is the limit of the sequence* $a_1b_1, a_2b_2, a_3b_3, \cdots$.

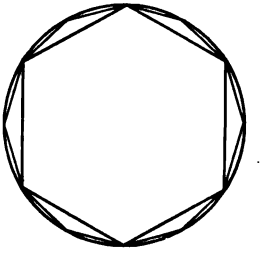


But by definition the limit of $a_1b_1, a_2b_2, a_3b_3, \cdots$ is the product ab. Hence we have the

THEOREM. *The area af a rectangle is the product of its base and altitude.*

530. Incommensurable Ratios Related to Two Circles. In a circle inscribe a sequence $P_1, P_2, P_3, \cdots$ of regular polygons, each having twice the number of sides of the one preceeding it.

Let the perimeters of these polygons be $p_1, p_2, p_3, \cdots$ and their areas $A_1, A_2, A_3, \cdots$. Then the length c of the circle is defined to be the limit of the sequence $p_1, p_2, p_3, \cdots$ and its area A is defined to be the limit of $A_1, A_2, A_3, \cdots$.



The sequence of polygons thus inscribed is called an *approximating sequence of polygons.*

That these sequences are *increasing* and *bounded* is obvious from the figure.

531. THEOREM. *The lengths of two circles are in the same ratio as their radii, and their areas are in the same ratio as the squares of their radii.*

Proof: Let the radii of the two circles be r and r'. Denote the ratio $r' : r$ by k. Then $r' = kr$.

Inscribe in one circle an approximating sequence of polygons with perimeters $p_1, p_2, p_3, \cdots$ and areas $A_1, A_2, A_3, \cdots$. In the other circle inscribe a sequence of similar polygons. By §§ 476, 477, the perimeters of the latter are $kp_1, kp_2, kp_3, \cdots$ and their areas are $k^2A_1, k^2A_2, k^2A_3, \cdots$.

By § 528, if the limit of $p_1, p_2, p_3, \cdots$ is c and the limit of $A_1, A_2, A_3, \cdots$ is A, then the limits of $kp_1, kp_2, kp_3, \cdots$, and $k^2A_1, k^2A_2, k^2A_3, \cdots$ are kc and k^2A, respectively.

That is, the ratio of the lengths of the circles is

$$\frac{kc}{c} = k = \frac{r'}{r} \text{ and the ratio of their areas is } \frac{k^2A}{A} = k^2 = \frac{r'^2}{r^2}.$$

If regular circumscribed polygons are used, we obtain *decreasing* sequences with the same limits as above for the perimeters and the areas.

EXERCISES

1. In addition to those which are found in the text give other examples of infinite sequences which do not determine definite numbers.

2. Give two increasing sequences which determine the number 3. Show that the theorem of § 520 applies and proves that these sequences determine the same number.

3. Give two decreasing sequences each of which determines the number 5. Apply § 521 to show that these sequences determine the same number.

4. State fully the relation between a bounded increasing sequence and the number determined by it. State also the relations between a bounded decreasing sequence and the number determined by it.

5. State fully what is meant by "a limit of a sequence" both for increasing and decreasing sequences.

6. Given two incommensurable segments AB and CD. Lay off on the line AB a *decreasing* sequence of segments, each of which is commensurable with CD, such that the limit of the sequence is the segment AB.

7. If $a_1, a_2, a_3, \cdots$ is an increasing sequence defining the number 4, prove that $3a_1, 3a_2, 3a_3, \cdots$ defines the number $3 \times 4 = 12$.

8. If a_1, a_2, a_3, and $b_1, b_2, b_3, \cdots$ are increasiug sequences defining the numbers 3 and 5, show that the sequence $a_1b_1, a_2b_2, a_3b_3, \cdots$ defines the number 15.

9. In the same manner as in § 524 define the ratio of two incommensurable arcs. Show that this ratio is independent of the sequence of units of measurement used, so long as the limit of this sequence is zero.

10. Treat the ratio of two incommensurable angles in a manner similar to the treatment of arcs in the preceding exercises.

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